

LINEAR ORDERS: WHEN EMBEDDABILITY AND EPIMORPHISM AGREE

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ABSTRACT. When a linear order has an order preserving surjection onto each of its suborders we say that it is *strongly surjective*. We prove that the set of countable strongly surjective linear orders is a $\bar{D}_2(\Pi_1^1)$ -complete set. Using hypotheses beyond ZFC, we prove the existence of uncountable strongly surjective orders.

CONTENTS

1. Introduction	1
1.1. About notations	3
2. Strong surjectivity	4
2.1. Defining epimorphisms	6
2.2. Operations on strongly surjective orders	8
3. Bounding the complexity of StS from below	10
3.1. Basic hardness	11
3.2. Powers of \mathbb{Z}	12
3.3. The lower bound	14
4. Bounding the complexity of StS from above	16
4.1. Some effective facts	16
4.2. Uniformizations for epimorphisms	18
5. Looking for uncountable strongly surjective orders	23
5.1. Classical examples are not strongly surjective	23
5.2. Beyond ZFC	28
5.3. Some directions for further research	30
Acknowledgments	31
References	32

1. INTRODUCTION

There are two natural ways of comparing a pair of linear orders L and M : embeddability and epimorphism. We write $L \leq_i M$ when there is an order preserving injection, also called an *embedding*, from L to M . Similarly, $L \leq_s M$ stands for the existence of an order preserving surjection, also called an *epimorphism*, from M onto L . The equivalence relation associated to \leq_s is written \equiv_s .

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Using the axiom of choice, $L \leq_s M$ implies $L \leq_i M$, but the embeddability relation \leq_i is in general weaker than the relation \leq_s induced by epimorphisms. For example, the ordinal number ω embeds into $\omega + 1$, but there is no epimorphism from $\omega + 1$ onto ω .

There are however linear orders M for which the relations $L \leq_i M$ and $L \leq_s M$ turn out to be equivalent. The ordinals satisfying this property have been characterized in [CCM15] (see Theorem 1.2 below). The aim of this article is to study the class of orders M for which the two notions coincide.

For the purpose of this paper, when talking about a linear order, we will always assume that it is non-empty; in particular, if no contrary mention is given, when a linear order is written as a sum $\sum_{i \in I} L_i$, all the summands are assumed to be non-empty.

We are now ready to give our main definition.

Definition 1.1. A linear order M is *strongly surjective* if, for any linear order L , $L \leq_i M$ implies $L \leq_s M$. Equivalently, if M surjects order-preservingly onto each of its suborders.

The following characterization of strongly surjective ordinals is Corollary 29 of [CCM15].

Theorem 1.2. *An ordinal is strongly surjective if and only if it is a finite multiple of an indecomposable countable ordinal, that is, if it is of the form $\omega^\alpha m$, for some $\alpha < \omega_1$ and $m > 0$.*

The rationals are also strongly surjective: indeed by [CCM15, Proposition 16(1)] $L \leq_s \mathbb{Q}$ for every countable linear order L . Up to \equiv_s , \mathbb{Q} is the only countable non-scattered strongly surjective order (recall that L is *scattered* if $\mathbb{Q} \not\leq_i L$): see Proposition 2.2 below.

Our main result is the following classification of the descriptive complexity of the set of countable strongly surjective linear orders:

Theorem 1.3. *The set of countable strongly surjective orders is $\check{D}_2(\Pi_1^1)$ -complete.*

Here $\check{D}_2(\Pi_1^1)$ is the class of sets which are union of an analytic and a coanalytic set. The set we are interested in belongs to this class because the set of scattered strongly surjective orders is Π_1^1 , while the set of non-scattered strongly surjective orders is Σ_1^1 . In fact they are both complete in their respective classes (Corollary 4.17 and Proposition 3.3).

Our proof of the upper bound for scattered strongly surjective orders makes an essential use of both effective descriptive set theory and the fact that \leq_s is a well quasi-order on the countable linear orders. The latter is the main theorem of [La79] and [CCM15].

Even if the study of the first two levels of the projective hierarchy is a long-standing topic, examples of sets that are true Δ_2^1 (that is, Δ_2^1

but neither analytic nor coanalytic) are very rare. The interest in these sets has recently been rekindled by Fournier's study of the difference hierarchy of co-analytic sets ([F16]). However, as far as we know, **StS** is the first concrete example of a $\check{D}_2(\Pi_1^1)$ -complete set that is not made so by design.

Here is the plan of the paper.

In Section 2, we prove some basic properties of strongly surjective linear orders, and we present a useful way of defining epimorphisms by pieces, that we use throughout the paper.

We start studying the descriptive complexity of the set of countable strongly surjective linear orders in Section 3. The set **Lin** of all linear orders \leq_K on a subset K of \mathbb{N} is Polish as it is a closed subspace of $2^{\mathbb{N} \times \mathbb{N}}$. We then call **StS** the set of strongly surjective orders in **Lin**. Definition 1.1 immediately gives an upper bound, **StS** being indeed a Π_2^1 subset of **Lin**. We prove in this section that **StS** is $\check{D}_2(\Pi_1^1)$ -hard (Theorem 3.11). Our proof uses a study of the powers of \mathbb{Z} and we notably prove that \mathbb{Z}^K is strongly surjective for all countable K .

In Section 4 we show that for any countable scattered linear order K , there is a $\Delta_1^1(K)$ function that maps a linear order L to an epimorphism from K to L when it exists, and to the refusing symbol \perp otherwise (Theorem 4.16). As a corollary we get a $\check{D}_2(\Pi_1^1)$ definition of **StS** (Corollary 4.18). This completes the proof of Theorem 1.3.

Finally, Section 5 deals with uncountable linear orders. We first prove that many concrete (e.g. \mathbb{R} , $\mathbb{R} \setminus \mathbb{Q}$, their finite products, and also $\mathbb{R}^{\mathbb{N}}$, $\mathbb{Q}^{\mathbb{N}}$, 2^α for $\alpha < \omega_1$) are not strongly surjective, leaving open the problem of the provability in ZFC of the existence of an uncountable strongly surjective linear order. By contrast, we prove the existence of uncountable strongly surjective orders assuming either PFA (Theorem 5.13) or the existence of what we call a Baumgartner tree (Theorem 5.18). The latter hypothesis is connected to the principle \diamond^+ , and thus orthogonal to PFA.

We conclude by discussing some problems that remain open and suggest new lines of research.

1.1. About notations. Variable symbols K, L, M always stand for linear orders. L^* stands for the reverse of the linear order L . We call *equimorphism* the equivalence relation \equiv_i associated to \leq_i , and we use the symbol \simeq to denote isomorphism. The notation for operations such as sums and products on linear orders is standard; a reference is [R82].

Given an order (K, \leq_K) , and $p \in K$, define $(\leftarrow, p)_K$ as $\{n \in K \mid n \leq_K p\}$ and order it with the order induced by \leq_K . Define in a similar fashion the orders $(\leftarrow, p)_K$, $(p, q)_K$, $(p, q]_K$, $[p, q)_K$, $[p, q]_K$, $(p, \rightarrow)_K$ and $[p, \rightarrow)_K$. We allow the notation $[p, q]_K$ when $p = q$ as well, letting then $[p, p]_K = \{p\}$. All these sets will be called *intervals*.

A subset K' of K is *convex* when $x, y \in K'$ and $x \leq_K y$ imply $[x, y]_K \subseteq K'$ (so every interval is convex, but not all convex sets are intervals).

We call $\text{Emb}(L, K)$ the set of all embeddings from L to K , and $\text{Epi}(L, K)$ the set of all epimorphisms from K onto L .

2. STRONG SURJECTIVITY

We begin by stating some basic properties of strongly surjective orders.

Proposition 2.1.

- (1) A linear order L is strongly surjective if and only if L^* is.
- (2) If L is strongly surjective and $M \leq_i L \leq_s M$, then M is strongly surjective.
- (3) If L and M are strongly surjective and $L \equiv_i M$, then $L \equiv_s M$.

Proof. (1) is obvious. (2) Let $K \leq_i M$. Since $M \leq_i L$ and L is strongly surjective, there is an epimorphism $L \rightarrow K$. As there is also an epimorphism $M \rightarrow L$, this yields $K \leq_s M$. (3) follows from the definition of strongly surjective. \square

Part (3) of Proposition 2.1 states that in any class of equimorphism there is at most one \equiv_s -class of strongly surjective orders. However, not every class of equimorphism contains a strongly surjective order. Indeed, for an ordinal number α the classes of equimorphism, isomorphism and bi-epimorphism coincide. So if α is not of the form given by Theorem 1.2, its equimorphism class does not contain any strongly surjective order.

The results of [CCM15] easily yield the following characterizations of countable strongly surjective linear orders that are not scattered:

Proposition 2.2. *Let L be a countable non-scattered linear order. The following are equivalent:*

- (i) L is strongly surjective;
- (ii) $\mathbb{Q} \leq_s L$;
- (iii) L has no initial or final segment which is scattered.

Proof. The equivalence of (i) and (ii) follows because all countable non-scattered linear orders are equimorphic and \mathbb{Q} is strongly surjective. By the above observation L is strongly surjective if and only if $L \equiv_s \mathbb{Q}$, which in turn is equivalent to $\mathbb{Q} \leq_s L$ because $L \leq_s \mathbb{Q}$ for every countable L by [CCM15, Proposition 16(1)].

The equivalence of (ii) and (iii) follows from [CCM15, Proposition 17]. \square

Definition 2.3. Given a linear order L without maximum, the *cofinality* of L , denoted $\text{cof}(L)$, is the smallest ordinal number α such that there exists an increasing function $\alpha \rightarrow L$ unbounded above in L .

Similarly, for a linear order L without minimum, the *coinitiality* of L , denoted $\text{coi}(L)$, is the reverse α^* of the smallest ordinal α such that there exists an increasing function $\alpha^* \rightarrow L$ unbounded below in L . Equivalently, $\text{coi}(L) = (\text{cof}(L^*))^*$.

Recall that L is *short* means that $\omega_1 \not\leq_i L$ and $\omega_1^* \not\leq_i L$.

Recall the following fact ([CCM15, Fact 14(5)]):

Proposition 2.4. *If K and L have no maximum and $K \leq_s L$, then $\text{cof}(K) = \text{cof}(L)$. Similarly, if K, L have no minimum and $K \leq_s L$, then $\text{coi}(K) = \text{coi}(L)$.*

Proposition 2.5.

- (1) *If a strongly surjective order has a minimum, then it is a well-order. If it has a maximum, then it is the reverse of a well-order.*
- (2) *A strongly surjective linear order that is not an ordinal has coinitiality ω^* . Similarly, a strongly surjective linear order that is not the reverse of an ordinal has cofinality ω .*
- (3) *Every strongly surjective linear order is short.*
- (4) *The cardinality of a strongly surjective linear order cannot exceed the continuum*

Proof. (1) If L is a strongly surjective order with a minimum and K is a non-empty subset of L , then K must have a minimum, otherwise $K \leq_s L$ would be impossible. Similarly for the maximum.

(2) If L is an ill-founded strongly surjective order, then $\omega^* \leq_i L$ and so $\omega^* \leq_s L$. It also follows that L does not have a minimum. So $\text{coi}(L) = \omega^*$ by Proposition 2.4. Similarly for the cofinality.

(3) By Proposition 2.4, (1) and (2) any suborder of a strongly surjective order L must have either a maximum or cofinality ω . Therefore $\omega_1 \not\leq_i L$. Similarly $\omega_1^* \not\leq_i L$.

(4) follows from (3) and a classical theorem of Urysohn's ([R82, Theorem 9.28]) about short linear orders. \square

It is useful to give a name to the orders satisfying the necessary conditions for strong surjectivity given in the first two items of Proposition 2.5.

Definition 2.6. A linear order L is *admissible* if the following conditions hold:

- (1) L has a minimum or it has coinitiality ω^* ;
- (2) L has a maximum or it has cofinality ω .

So an order is short if and only if it and all of its suborders are admissible.

2.1. Defining epimorphisms. Given non-empty convex subsets K_0 and K_1 of K , say that $K_0 \leq K_1$ when for all $x \in K_0$ and $y \in K_1$ we have $x \leq_K y$; similarly define $K_0 < K_1$ if for all $x \in K_0$ and all $y \in K_1$ one has $x <_K y$. Say that K_0 and K_1 are *adjacent* if $K_0 \leq K_1$ and there is no $x \in K$ satisfying $K_0 < \{x\} < K_1$. Say they are *connected* when $K_0 \leq K_1$ but $K_0 \not\leq K_1$ (so that they share an element).

An epimorphism can be defined on a covering by convex sets.

Definition 2.7. We say that a family of non-empty convex sets $(K_i)_{i \in I}$ of an order K is *nice* if and only if the index set I is an interval of \mathbb{Z} , the family $(K_i)_{i \in I}$ is unbounded above and below in K and for all $i \in I$, $K_i \leq K_{i+1}$ holds.

We say that $(K_i)_{i \in I}$ is a *nice covering* of K if it is a covering of K by a nice family.

Lemma 2.8 (Definition by pieces). *Suppose we have $(K_i)_{i \in I}$ a nice family of convex subsets of K and $(L_i)_{i \in I}$ a nice covering of L satisfying that for any $i \in I$ when K_i and K_{i+1} are not adjacent then L_i has a maximum or L_{i+1} has a minimum, and if K_i and K_{i+1} are connected, so are L_i and L_{i+1} .*

If for all $i \in I$, $L_i \leq_s K_i$ holds, then $L \leq_s K$.

Proof. Take $\sigma_i \in \text{Epi}(L_i, K_i)$ and whenever K_i and K_{i+1} are not adjacent let l_i be the maximum of L_i if it exists, or the minimum of L_{i+1} otherwise. Define then the map $\sigma : K \rightarrow L$ as follows.

$$\sigma(x) = \begin{cases} \sigma_i(x) & \text{if } x \in K_i \text{ for some } i \in I \\ l_i & \text{if } K_i < \{x\} < K_{i+1} \text{ for some } i \in I \end{cases}$$

We defined σ on every K_i and on the convex sets between K_i and K_{i+1} , so on all of K . Let us first check that it is well-defined. Suppose x is in $K_i \cap K_{i+1}$ for some $i \in I$. Then, since $K_i \leq K_{i+1}$, we have $K_i \cap K_{i+1} = \{x\}$, so that $x = \max K_i = \min K_{i+1}$ connects the two intervals. The hypothesis gives that $\max L_i = \min L_{i+1}$, and as the maps σ_i and σ_{i+1} are epimorphisms we have

$$\sigma_i(x) = \sigma_i(\max K_i) = \max L_i = \min L_{i+1} = \sigma_{i+1}(\min K_{i+1}) = \sigma_{i+1}(x),$$

so σ is indeed well-defined. Since the maps σ_i are epimorphisms and the sets L_i form a nice covering of L , we finally have $\sigma \in \text{Epi}(L, K)$. \square

In the above proof, we say that σ is *defined by pieces*. Some specific operations come in handy to define epimorphisms by pieces.

Definition 2.9. Given K, L linear orders, $\sigma \in \text{Epi}(L, K)$ and $l \in L$, we denote σ^l the following epimorphism:

$$\sigma^l : K \longrightarrow (\leftarrow, l]_L$$

$$k \longmapsto \begin{cases} \sigma(k) & \text{if } \sigma(k) \leq_L l \\ l & \text{otherwise.} \end{cases}$$

Similarly we define

$$\begin{aligned}\sigma_l : K &\longrightarrow [l, \rightarrow)_L \\ k &\longmapsto \begin{cases} \sigma(k) & \text{if } \sigma(k) \geq_L l \\ l & \text{otherwise.} \end{cases}\end{aligned}$$

Given $l \leq_L l'$, we also define

$$\begin{aligned}\sigma_l^{l'} : K &\longrightarrow [l, l']_L \\ k &\longmapsto \begin{cases} l & \text{if } \sigma(k) < l \\ \sigma(k) & \text{if } l \leq \sigma(k) \leq l' \\ l' & \text{if } l' < \sigma(k). \end{cases}\end{aligned}$$

Proposition 2.10 (Family mash). *Given linear orders K and L , with L admissible, if there is a nice family $(K_i)_{i \in I}$ of K such that $L \leq_s K_i$ holds for all $i \in I$, then we have $L \leq_s K$.*

Proof. We may assume that L is not a singleton, and so no K_i is a singleton. Moreover, we can suppose that I has more than one element, otherwise if $I = \{i\}$ then $K_i = K$ (because K_i is convex and unbounded) and we are done.

For each $i \in I$ fix $\sigma_i \in \text{Epi}(L, K_i)$. We want to define $\sigma \in \text{Epi}(L, K)$ by pieces.

Suppose first that I is finite, and let i and j be its minimum and maximum. Notice that (K_i, K_j) is nice. On the L side, take any $l \in L$ and consider the nice connected covering $(\leftarrow, l]_L, [l, \rightarrow)_L$. Then we can use Lemma 2.8, with $(\sigma_i)^l$ and $(\sigma_j)_l$ witnessing $(\leftarrow, l]_L \leq_s K_i$ and $[l, \rightarrow)_L \leq_s K_j$ respectively, to define σ .

From now on, we suppose that I is infinite. There are four cases.

Case 1: When L has a minimum l_0 and a maximum l_1 , choose $i \in I$ different from the minimum and maximum of I (at most one of the extrema exists), and let $K^- = \{k \in K \mid \{k\} < K_i\}$ and $K^+ = \{k \in K \mid K_i < \{k\}\}$. The nice covering (K^-, K_i, K^+) of K and the connected covering $(\{l_0\}, L, \{l_1\})$ of L allow the definition by pieces of σ .

Case 2: When L has a minimum \hat{l} and no maximum, we need to distinguish two subcases.

If I has a maximum i we consider the nice covering $(K \setminus K_i, K_i)$ and the connected covering $(\{\hat{l}\}, L)$.

If instead I has no maximum, by admissibility of L let $\{l_n\}_{n \in \mathbb{N}}$ be strictly increasing and cofinal in L . Fix $i \in I$ and let $K'_i = K_i \cup \{k \in K \mid \{k\} < K_i\}$. Now consider the nice family $(K'_i, K_{i+1}, K_{i+2}, \dots)$ and the nice connected covering

$$([\hat{l}, l_i]_L, [l_i, l_{i+1}]_L, [l_{i+1}, l_{i+2}]_L, \dots).$$

Using $(\sigma_i)^{l_i}$ and $(\sigma_j)^{l_j}$ for $j > i$ we again get the definition by pieces of σ .

Case 3: When L has a maximum and no minimum, we can just mirror the previous case.

Case 4: When L has no extrema, we look for a connected nice covering $(L_i)_{i \in I}$ to match $(K_i)_{i \in I}$. Take $(l_i)_{i \in \mathbb{Z}}$ strictly increasing, coinital and cofinal in L . If $I = \mathbb{Z}$ then take $([l_i, l_{i+1}]_L)_{i \in I}$. If I has a minimum j , take $L_j = (\leftarrow, l_j]_L$ and $([l_i, l_{i+1}]_L)_{i \geq j}$. If I has a maximum j , take $([l_i, l_{i+1}]_L)_{i < j}$ and $L_j = [l_j, \rightarrow)_L$. In any case, we can use the appropriate $(\sigma_i)^{l_i}$, $(\sigma_j)^{l_j}$, and $(\sigma_j)^{l_j}$ to define σ by pieces. \square

In the above proof we say that we *mash* the family $(K_i)_{i \in I}$ onto L .

Corollary 2.11. *For K and L admissible, we have $L \leq_s LK$.*

Proof. By admissibility of K , take $(k_i)_{i \in I}$ increasing, coinital and cofinal in K for some I which is an interval in \mathbb{Z} . Mash the nice family $(L \times \{k_i\})_{i \in I}$ of LK onto L using Proposition 2.10. \square

2.2. Operations on strongly surjective orders. In general, the sum of strongly surjective orders is not strongly surjective: consider for example a countable ordinal whose Cantor normal form has two summands and use Theorem 1.2. We now show instead that the product of two strongly surjective orders is still strongly surjective, and that the left-quotient of a strongly surjective order by a scattered order is also strongly surjective. First note the following.

Proposition 2.12. *Let I be any order and for each $i \in I$, let L_i be a strongly surjective order. Then $L = \sum_{i \in I} L_i$ is strongly surjective if and only if for every non-empty $J \subseteq I$ there is an epimorphism from L onto $L_J = \sum_{j \in J} L_j$.*

Proof. If L is strongly surjective, then it must admit an epimorphism onto its suborder L_J for any non-empty $J \subseteq I$.

Conversely, suppose there is an epimorphism $\psi_J : L \rightarrow L_J$ for any non-empty $J \subseteq I$. Let K be a suborder of L and let J be the set of indices j such that K intersects L_j in a non-empty set K_j , so that $K = \sum_{j \in J} K_j$. Since each L_j is strongly surjective, let $\varphi_j : L_j \rightarrow K_j$ be an epimorphism. These induce an epimorphism $\varphi : L_J \rightarrow K$. Then $\varphi \circ \psi_J : L \rightarrow K$ is an epimorphism. \square

This yields the following simple examples of strongly surjective orders.

Example 2.13. Let γ, δ be countable ordinals and $n, m > 0$. Then $(\omega^\gamma n)^*$, $\omega^\delta m$ and $(\omega^\gamma n)^* + \omega^\delta m$ are strongly surjective.

Proof. The fact that $(\omega^\gamma n)^*, \omega^\delta m$ are strongly surjective is a consequence of Theorem 1.2 and Proposition 2.1(1). So by Proposition 2.12 it is enough to show $(\omega^\gamma n)^* \leq_s (\omega^\gamma n)^* + \omega^\delta m$ and $\omega^\delta m \leq_s (\omega^\gamma n)^* + \omega^\delta m$, which can be done by a definition by pieces using the presence of an extremum in the range. \square

Corollary 2.14. *If L and M are strongly surjective, then LM is strongly surjective. In particular L^n is strongly surjective for all $n \in \mathbb{N}$.*

Proof. By Proposition 2.12, it is enough to show $LK \leq_s LM$ for any suborder K of M . Let $\varphi : M \rightarrow K$ be an epimorphism and for $k \in K$ let $M_k = \varphi^{-1}(\{k\})$. As M is strongly surjective, each M_k must be admissible. Since L is also admissible by Proposition 2.5, Corollary 2.11 implies that there is an epimorphism $\varphi_k : LM_k \rightarrow L$ for every $k \in K$. Gluing together these epimorphisms yields $LK \leq_s LM$. \square

Strongly surjective orders are not closed under infinite products (ordered lexicographically), as we will show in Section 5.

Lemma 2.15. *If L is a scattered linear order and $0 < n < m$, then $Lm \not\leq_i Ln$.*

Proof. We first show the special case $L2 \not\leq_i L$, which is actually [Mon06, Lemma 1.17]. Notice that if $L2 \leq_i L$ then an easy induction shows that $Ln \leq_i L$ for any n . We show that under this hypothesis L is not scattered. To this end we recursively define for every $s \in 2^{<\mathbb{N}}$ a subset L_s of L which is isomorphic to L and a point $x_s \in L$. Start with $L_\emptyset = L$. Assuming that $L_s \simeq L$ and $L_s \subseteq L$, since $L3 \leq_i L_s$, pick a point x_s in the middle copy of L embedded in L_s and let L_{s0} and L_{s1} be the left and right copies of L embedded in L_s . Then $\{x_s\}_{s \in 2^{<\mathbb{N}}}$ is a dense suborder of L , and so L is not scattered.

Now assume $0 < n < m$ and $Lm \leq_i Ln$. Again inductively one can show that $L(m + k(m - n)) \leq_i Ln$ for all k . If k is large enough we have $m + k(m - n) \geq 2n$ and hence $Ln2 \leq_i Ln$, which by the above implies that Ln is not scattered. Thus L is not scattered. \square

Proposition 2.16. *If L is scattered and LK is strongly surjective, then K is strongly surjective.*

Proof. Let $J \leq_i K$ and fix an epimorphism $\varphi : LK \rightarrow LJ$. Define the relation $R \subseteq K \times J$ by letting $kRj \Leftrightarrow \varphi(L \times \{k\}) \cap (L \times \{j\}) \neq \emptyset$. If $k \in K$ we denote by R_k the vertical section $\{j \in J \mid kRj\}$. Similarly, for $j \in J$, R^j is the horizontal section $\{k \in K \mid kRj\}$. Notice that:

- all sections are non-empty (i.e. the domain of R is K and its range is J);
- all sections are convex subsets of the respective linear order;
- $|R_k| \leq 3$ for each $k \in K$ (by Lemma 2.15).

To define an epimorphism $\psi : K \rightarrow J$, we only need to define a surjection ψ that satisfies $kR\psi(k)$ for all $k \in K$. Given $k \in K$ we distinguish several cases.

- (a) If $|R_k| = 1$ and j is the unique element of R_k set $\psi(k) = j$.
- (b) If there is j (necessarily unique, by Lemma 2.15) such that $L \times \{j\} \subseteq \varphi(L \times \{k\})$, then set $\psi(k) = j$; note that if $|R_k| = 3$ then k satisfies this case.
- (c) So it remains to define ψ on the set H of those k such that $|R_k| = 2$, but do not fall in case (b). Consider I a maximal \leq_K -convex subset of H that is embeddable in \mathbb{Z} : I is contained in a \mathbf{c}_F -condensation class C of K ; see [R82, §4.2]. So I has order type finite, ω , ω^* or $\omega^* + \omega$. We need to define ψ on each such I .
 - (c1) Suppose first that I has n elements $k_0 < \dots < k_{n-1}$. Consequently, $\bigcup_{r=0}^{n-1} R_{k_r}$ consists of $n+1$ consecutive points of J , say $j_0 < \dots < j_n$, so that $R_{k_r} = \{j_r, j_{r+1}\}$.
 - (c1a) If $k_0 = \min C$, set $\psi(k_r) = j_{r+1}$ for $0 \leq r \leq n-1$. Notice that in this case, since j_0 does not witness that case (b) applies to k_0 , R^{j_0} consists of k_0 and an infinite convex set with supremum k_0 : then for all but at most one $k \in R^{j_0} \setminus \{k_0\}$ we have $\psi(k) = j_0$ by case (a).
 - (c1b) If $k_0 \neq \min C$ but $k_{n-1} = \max C$, then let $\psi(k_r) = j_r$ for $0 \leq r \leq n-1$. An argument similar to the one used in case (c1a) shows that in this case $\psi(k) = j_n$ for some k .
 - (c1c) If neither k_0 is the first element of C , nor k_{n-1} is the last element of C , let k' be the immediate predecessor of k_0 and k'' be the immediate successor of k_{n-1} in K . Thus $k'Rj_0$, $k''Rj_n$ (because neither k_0 nor k_{n-1} satisfy the condition of case (b)) and both $\psi(k')$ and $\psi(k'')$ have been defined according to cases (a) or (b). Notice that $\psi(k')$ and $\psi(k'')$ cannot have both been defined according to clause (b), with values different from j_0, j_n , respectively, as in this case one would have $L(n+3) \leq_i L(n+2)$, contradicting Lemma 2.15. This implies that either $\psi(k') = j_0$ or $\psi(k'') = j_n$. If $\psi(k') = j_0$, let $\psi(k_r) = j_{r+1}$; otherwise, let $\psi(k_r) = j_r$.
 - (c2) If I has order type ω , ω^* or $\omega^* + \omega$, then $\bigcup_{k \in I} R_k$ has the same order type and we can define ψ on I as any order preserving surjection onto $\bigcup_{k \in I} R_k$.

By construction, $\psi : K \rightarrow J$ is order preserving and surjective. \square

3. BOUNDING THE COMPLEXITY OF **StS** FROM BELOW

The closure properties of Subsection 2.2 allow to build several examples of strongly surjective linear orders. We present here other kinds of examples allowing to obtain some hardness results.

First we make our formal setting precise. We call Lin the subset of $2^{\mathbb{N} \times \mathbb{N}}$ consisting of all linear orders \leq_K on a subset $K = \text{dom}(\leq_K)$ of \mathbb{N} . By definition it is a Polish subspace of $2^{\mathbb{N} \times \mathbb{N}}$. To avoid heavy notations, when there is no possible confusion we just write K for the pair (K, \leq_K) .

When we work with elements of Lin we fix recursive copies of \mathbb{N} and \mathbb{Q} , denoted respectively by ω and η . Moreover we assume a fixed way of implementing sums (finite or infinite) and products as recursive (and hence continuous) operations which produce new elements of Lin .

Remark 3.1. In the literature most often people work with LO , the space of all total orders on the domain \mathbb{N} . The downside of LO is the absence of finite orders, which we need for the main result in Section 4. That is why we deal with Lin . However, for the classification results on strongly surjective orders the two settings are equivalent. Indeed, denote by Fin the Σ_2^0 set of finite orders in Lin , and notice that there are continuous functions $\text{LO} \rightarrow \text{Lin} \setminus \text{Fin}$ and $\text{Lin} \setminus \text{Fin} \rightarrow \text{LO}$ that preserve order types. If Γ is a pointclass that includes Σ_2^0 and is closed under finite unions and continuous preimages, and the set of strongly surjective orders in LO belongs to Γ , then $\text{StS} \setminus \text{Fin} \in \Gamma$, so $\text{StS} \in \Gamma$. Conversely, if $\text{StS} \in \Gamma$, one has that the strongly surjective orders as a subset of LO are in Γ as well.

3.1. Basic hardness. Let Scat and WO be the subsets of Lin consisting of the scattered countable linear orders and of the countable well-orders. It is well-known that both Scat and WO are Π_1^1 and Π_1^1 -complete.

Proposition 3.2. *The sets StS , $\text{StS} \cap \text{Scat}$ and $\text{StS} \cap \text{WO}$ are Π_1^1 -hard.*

Proof. Let $g : \text{Lin} \rightarrow \text{Lin}$ be defined by $g(L) = (1 + L)\omega$. Using Proposition 2.5 and Theorem 1.2, as $g(L)$ has a minimal element for any L , we have $g(L) \in \text{StS}$ if and only if $g(L) \in \text{WO}$. Since $g(L) \in \text{WO}$ if and only if $L \in \text{WO}$, we have that g reduces WO to StS , to $\text{StS} \cap \text{Scat}$, and to $\text{StS} \cap \text{WO}$ as well. \square

We now consider the set of countable strongly surjective linear orders that are non-scattered.

Proposition 3.3. *The set StS is Σ_1^1 -hard, and the set of non-scattered strongly surjective orders is Σ_1^1 and Σ_1^1 -complete.*

Proof. Let $f : \text{Lin} \rightarrow \text{Lin}$ be defined by $f(L) = \eta + L\omega$. As $f(L)$ is non-scattered for all L , we have $f(L) \in \text{StS}$ if and only if it has no scattered initial nor final segments by Proposition 2.2. But $f(L)$ never has a scattered initial segment, and it has a scattered final segment if and only if L itself is scattered. So finally f reduces $\text{Lin} \setminus \text{Scat}$ to StS , and even to the set of non-scattered strongly surjective countable linear orders, which are consequently Σ_1^1 -hard.

The fact that $\text{StS} \setminus \text{Scat}$ is Σ_1^1 follows from the characterization of Proposition 2.2.(ii). \square

3.2. Powers of \mathbb{Z} . The main new ingredient needed for the lower bound is a general version of the exponentiation with base \mathbb{Z} . There are two definitions of \mathbb{Z}^α for α an ordinal number. The first one is by ordinal induction (see [R82, Definition 5.34]), while the second ([R82, Definition 5.35]) is a direct set theoretic definition, and it can actually be used as a definition of \mathbb{Z}^K for K any linear order. As pointed out in [R82, Exercise 5.36(1)], the two definitions coincide when K is a well-order.

We first recall the definition by ordinal induction.

Definition 3.4.

- (1) $\mathbb{Z}^0 = 1$,
- (2) $\mathbb{Z}^{\alpha+1} = \mathbb{Z}^\alpha \omega^* + \mathbb{Z}^\alpha + \mathbb{Z}^\alpha \omega$,
- (3) $\mathbb{Z}^\alpha = (\sum_{\beta < \alpha} \mathbb{Z}^\beta \omega)^* + 1 + \sum_{\beta < \alpha} \mathbb{Z}^\beta \omega$ if α is a limit ordinal.

The following equalities will be useful.

Proposition 3.5. *For any α and $\beta < \alpha$, we have*

$$\mathbb{Z}^\alpha = \left(\sum_{\gamma < \alpha} \mathbb{Z}^\gamma \omega \right)^* + 1 + \sum_{\gamma < \alpha} \mathbb{Z}^\gamma \omega = \left(\sum_{\beta \leq \gamma < \alpha} \mathbb{Z}^\gamma \omega \right)^* + \sum_{\beta \leq \gamma < \alpha} \mathbb{Z}^\gamma \omega.$$

Proof. To prove the first equality we argue by induction on α . The cases $\alpha = 0$ and α limit are immediate from the definition. For the successor case we have

$$\begin{aligned} \mathbb{Z}^{\alpha+1} &= \mathbb{Z}^\alpha \omega^* + \mathbb{Z}^\alpha + \mathbb{Z}^\alpha \omega \\ &= \mathbb{Z}^\alpha \omega^* + \left(\sum_{\gamma < \alpha} \mathbb{Z}^\gamma \omega \right)^* + 1 + \sum_{\gamma < \alpha} \mathbb{Z}^\gamma \omega + \mathbb{Z}^\alpha \omega \\ &= \left(\sum_{\gamma < \alpha+1} \mathbb{Z}^\gamma \omega \right)^* + 1 + \sum_{\gamma < \alpha+1} \mathbb{Z}^\gamma \omega, \end{aligned}$$

where in the central step we use the induction hypothesis.

The second equality can be proved applying the first one to β , using $(\mathbb{Z}^\beta \omega)^* + \mathbb{Z}^\beta \omega = \mathbb{Z}^\beta \mathbb{Z} = \mathbb{Z}^{\beta+1}$. \square

Proposition 3.6. *For any countable ordinal α and natural number $m > 0$, the order $\mathbb{Z}^\alpha m$ is strongly surjective.*

Proof. Since finite linear orders are strongly surjective by Theorem 1.2, by Corollary 2.14 it suffices to show that each \mathbb{Z}^α is strongly surjective. Proceed by induction on α . When $\alpha = 0$ we get the singleton linear order. Notice that \mathbb{Z} is strongly surjective by Example 2.13, so that Corollary 2.14 handles the successor step because $\mathbb{Z}^{\alpha+1} = \mathbb{Z}^\alpha \mathbb{Z}$.

Suppose now that δ is limit and that \mathbb{Z}^γ is strongly surjective for all $\gamma < \delta$. By Corollary 2.14, so are $\mathbb{Z}^\gamma \omega$ and $\mathbb{Z}^\gamma \omega^*$. Recall that, by

Proposition 3.5, \mathbb{Z}^δ can be written as a sum over the index set $I = \delta^* + \delta$: $\mathbb{Z}^\delta = (\sum_{\gamma < \delta} \mathbb{Z}^\gamma \omega)^* + \sum_{\gamma < \delta} \mathbb{Z}^\gamma \omega$.

First we show that if $1 \leq \beta_0 < \beta_1 \leq \delta$ and H is a non-empty subset of δ with $\sup H < \beta_0$, then

$$(1) \quad \sum_{\gamma \in H} \mathbb{Z}^\gamma \omega \leq_s \sum_{\beta_0 \leq \gamma < \beta_1} \mathbb{Z}^\gamma \omega.$$

If $\beta_1 = \rho + 1$ is a successor ordinal, $(\mathbb{Z}^{\beta_0} \omega, \mathbb{Z}^\rho \omega)$ is a nice family in $\sum_{\beta_0 \leq \gamma < \beta_1} \mathbb{Z}^\gamma \omega$ and we can mash $\sum_{\beta_0 \leq \gamma < \beta_1} \mathbb{Z}^\gamma \omega$ onto $\sum_{\gamma \in H} \mathbb{Z}^\gamma \omega$, since $\sum_{\gamma \in H} \mathbb{Z}^\gamma \omega \leq_i \mathbb{Z}^{\beta_0} \omega \leq_i \mathbb{Z}^{\beta_0} \omega \leq_i \mathbb{Z}^\rho \omega$. If β_1 is limit, let ρ_n be an increasing cofinal sequence in β_1 , with $\rho_0 = \beta_0$: we can mash the nice family $(\mathbb{Z}^{\rho_n} \omega)_{n \in \mathbb{N}}$ onto $\sum_{\gamma \in H} \mathbb{Z}^\gamma \omega$. So, in either case we get (1).

Take now a non-empty subset J of I : it determines two subsets $J^-, J^+ \subseteq \delta$ – one of them possibly empty – and a suborder $K = (\sum_{\gamma \in J^-} \mathbb{Z}^\gamma \omega)^* + \sum_{\gamma \in J^+} \mathbb{Z}^\gamma \omega$. We want to show that $K \leq_s \mathbb{Z}^\delta$, so that we can conclude the proof by applying Proposition 2.12. Set $K^- = (\sum_{\gamma \in J^-} \mathbb{Z}^\gamma \omega)^*$ and $K^+ = \sum_{\gamma \in J^+} \mathbb{Z}^\gamma \omega$.

First notice that we may suppose that both K^- and K^+ are non-empty. In fact if, for example, $K^- = \emptyset$, letting $\alpha = \min J^+$, we have

$$K = K^+ = \mathbb{Z}^\alpha + K^+ \leq_s (\mathbb{Z}^\alpha \omega)^* + K^+,$$

so that an epimorphism from \mathbb{Z}^δ onto the rightmost part gives by composition an epimorphism onto K .

Similarly, we may assume that both J^- and J^+ are unbounded in δ . Indeed, if $\alpha < \delta$ is an upper bound for, say, J^+ , we have

$$K \leq_s K^- + \mathbb{Z}^{\alpha+1} \leq_s K^- + \mathbb{Z}^{\alpha+1} \omega \leq_s K^- + \sum_{\alpha+2 \leq \gamma < \delta} \mathbb{Z}^\gamma \omega,$$

where in the first inequality we used the induction hypothesis, and in the last one (1).

Since $\mathbb{Z}^\delta = (\sum_{\beta \leq \gamma < \delta} \mathbb{Z}^\gamma \omega)^* + \sum_{\beta \leq \gamma < \delta} \mathbb{Z}^\gamma \omega$ for any $\beta < \delta$, it is enough to show that both $K^- \leq_s (\sum_{\beta \leq \gamma < \delta} \mathbb{Z}^\gamma \omega)^*$ and $K^+ \leq_s \sum_{\beta \leq \gamma < \delta} \mathbb{Z}^\gamma \omega$ hold for some β . To prove, for instance, the latter, let $\{\alpha_n\}_{n \in \mathbb{N}}$ be increasing and cofinal in J^+ , with $\alpha_0 = \min J^+$, and let $\{\beta_n\}_{n \in \mathbb{N}}$ be increasing and cofinal in δ , with $\forall n \in \mathbb{N} \alpha_{n+1} < \beta_n$. Then, by (1), there exist epimorphisms

$$\sum_{\beta_n \leq \gamma < \beta_{n+1}} \mathbb{Z}^\gamma \omega \rightarrow \sum_{\gamma \in J^+, \alpha_n \leq \gamma < \alpha_{n+1}} \mathbb{Z}^\gamma \omega$$

Gluing them together, one obtains an epimorphism from $\sum_{\beta_0 \leq \gamma < \delta} \mathbb{Z}^\gamma \omega$ onto K^+ . \square

Here is the set-theoretic definition of exponentiation with base \mathbb{Z} .

Definition 3.7. Let K be a linear order. For any map $\varphi : K \rightarrow \mathbb{Z}$, $\text{Supp}(\varphi)$ stands for the *support* of φ , that is $\text{Supp}(\varphi) = \{k \in K \mid \varphi(k) \neq 0\}$. The K -power of \mathbb{Z} , denoted by \mathbb{Z}^K , is the following order on $\{\varphi : K \rightarrow \mathbb{Z} \mid \text{Supp}(\varphi) \text{ is finite}\}$. If $\varphi, \psi : K \rightarrow \mathbb{Z}$ are maps with finite support let $\varphi \leq_{\mathbb{Z}^K} \psi$ if and only if $\varphi = \psi$ or $\varphi(k_0) <_{\mathbb{Z}} \psi(k_0)$ where $k_0 = \max\{k \in \text{Supp}(\varphi) \cup \text{Supp}(\psi) \mid \varphi(k) \neq \psi(k)\}$.

We now show that if K is countable but not a well-order then $\mathbb{Z}^K \equiv_s \mathbb{Q}$, and hence \mathbb{Z}^K is strongly surjective by Proposition 2.2.

Lemma 3.8. *For any linear orders K and L we have:*

- (1) $\mathbb{Z}^{K+L} \simeq \mathbb{Z}^K \mathbb{Z}^L$
- (2) *if K is countable and with no minimum then $\mathbb{Z}^K \simeq \mathbb{Q}$.*

Proof. (1) The bijection $\mathbb{Z}^{K+L} \rightarrow \mathbb{Z}^K \mathbb{Z}^L$, $\varphi \mapsto (\varphi|_K, \varphi|_L)$ is an isomorphism.

(2) Take K countable with no minimum, and suppose $\varphi <_{\mathbb{Z}^K} \psi$ holds for some φ and ψ in \mathbb{Z}^K . As K has no minimum pick $k_0 \in K$ that is strictly below every element of $\text{Supp}(\varphi) \cup \text{Supp}(\psi)$. Define φ^- , θ and ψ^+ , all in \mathbb{Z}^K , as follows.

$$\begin{aligned}\varphi^-(x) &= \begin{cases} -1 & \text{if } x = k_0 \\ \varphi(x) & \text{otherwise,} \end{cases} \\ \theta(x) &= \begin{cases} 1 & \text{if } x = k_0 \\ \varphi(x) & \text{otherwise,} \end{cases} \\ \psi^+(x) &= \begin{cases} 1 & \text{if } x = k_0 \\ \psi(x) & \text{otherwise.} \end{cases}\end{aligned}$$

We have

$$\varphi^- <_{\mathbb{Z}^K} \varphi <_{\mathbb{Z}^K} \theta <_{\mathbb{Z}^K} \psi <_{\mathbb{Z}^K} \psi^+,$$

so \mathbb{Z}^K is dense, countable, without extrema, giving $\mathbb{Z}^K \simeq \mathbb{Q}$. \square

Proposition 3.9. *If K is countable and not a well order then there is a countable ordinal α such that*

$$\mathbb{Z}^K \simeq \mathbb{Z}^\alpha \mathbb{Q}.$$

Hence $\mathbb{Z}^K \equiv_s \mathbb{Q}$.

Proof. To obtain $\mathbb{Z}^K \simeq \mathbb{Z}^\alpha \mathbb{Q}$ it suffices to use the previous lemma with the decomposition $K = \alpha + K'$ for α ordinal and K' without minimum. Since \mathbb{Z}^K is written as a \mathbb{Q} sum we have $\mathbb{Q} \leq_s \mathbb{Z}^K$, which yields $\mathbb{Z}^K \equiv_s \mathbb{Q}$ because K , and hence \mathbb{Z}^K , is countable. \square

3.3. The lower bound. We now prove that **StS** is hard for the class $\check{\mathbf{D}}_2(\Pi_1^1)$ of all sets that are the union of an analytic and a coanalytic set.

As we did for sums and products, we want to realize exponentiation with base \mathbb{Z} as an operation on Lin . Since in this case the details are less straightforward, we provide them:

Proposition 3.10. *There is a continuous (even recursive) function $\text{Lin} \rightarrow \text{Lin}$ mapping any K to an order $(\zeta^K, \leq_{\zeta^K})$ isomorphic to \mathbb{Z}^K .*

Proof. Fix $n \mapsto s_n$ and $n \mapsto ((n)_0, (n)_1)$ recursive enumerations of $\mathbb{N}^{<\mathbb{N}}$ and \mathbb{N}^2 respectively, as well as a recursive order \prec_ζ , whose domain is the whole \mathbb{N} , that is isomorphic to the strict part of $\leq_{\mathbb{Z}}$. For any $K \in \text{Lin}$ we define $(\zeta^K, \leq_{\zeta^K}) \in \text{Lin}$.

First, the domain ζ^K is the set of codes for pairs of sequences of the same length, the first with values in K , the second in $\mathbb{N} \setminus \{0\}$. This simulates the finite support. For convenience we require the sequences with values in K to be \leq_K -decreasing. Writing $\text{lh}(s)$ for the length of a sequence s :

$$\zeta^K = \left\{ n \in \mathbb{N} \mid \text{lh}(s_{(n)_0}) = \text{lh}(s_{(n)_1}) \wedge \forall i < \text{lh}(s_{(n)_0}) \ s_{(n)_1}(i) \neq 0 \right. \\ \left. \wedge \forall i < j < \text{lh}(s_{(n)_0}) \ s_{(n)_0}(j) <_K s_{(n)_0}(i) \right\}.$$

We now compare two codes of pairs of sequences on the first value on which they differ. We have to be careful because if the value differs on the first sequence, it means that the two sequences do not have the same support. Also, one sequence could extend the other. Formally, given $n, m \in \zeta^K$ we have $n \leq_{\zeta^K} m$ if and only if

- $n = m$;
- or
- for all $i < \min\{\text{lh}(s_{(n)_0}), \text{lh}(s_{(m)_0})\}$ we have

$$\forall \varepsilon \in \{0, 1\} (s_{(n)_\varepsilon}(i) = s_{(m)_\varepsilon}(i)),$$

and

$$(\text{lh}(s_{(n)_0}) < \text{lh}(s_{(m)_0}) \wedge 0 \prec_\zeta s_{(m)_1}(\text{lh}(s_{(n)_0}))) \vee \\ (\text{lh}(s_{(m)_0}) < \text{lh}(s_{(n)_0}) \wedge s_{(n)_1}(\text{lh}(s_{(m)_0})) \prec_\zeta 0);$$

or

- there exists $i < \min\{\text{lh}(s_{(n)_0}), \text{lh}(s_{(m)_0})\}$ satisfying

$$\forall j < i \ \forall \varepsilon \in \{0, 1\} (s_{(n)_\varepsilon}(j) = s_{(m)_\varepsilon}(j)),$$

and

$$(s_{(n)_0}(i) <_K s_{(m)_0}(i) \wedge 0 \prec_\zeta s_{(m)_1}(i)) \vee \\ (s_{(m)_0}(i) <_K s_{(n)_0}(i) \wedge s_{(n)_1}(i) \prec_\zeta 0) \vee \\ (s_{(n)_0}(i) = s_{(m)_0}(i) \wedge s_{(n)_1}(i) \prec_\zeta s_{(m)_1}(i)).$$

This is, given K , a recursive encoding of an order ζ^K isomorphic to \mathbb{Z}^K . \square

Theorem 3.11. *The set \mathbf{StS} is hard for the class $\check{D}_2(\Pi_1^1)$.*

Proof. First observe that the set

$$A = \{(K, L) \in \mathbf{Lin} \times \mathbf{Lin} \mid K \notin \mathbf{WO} \vee L \in \mathbf{WO}\}$$

is $\check{D}_2(\Pi_1^1)$ -complete: if $B = B_0 \cup B_1$, with f_0 reducing B_0 to $\mathbf{Lin} \setminus \mathbf{WO}$ and f_1 reducing B_1 to \mathbf{WO} , then (f_0, f_1) reduces B to A .

We now prove that \mathbf{StS} continuously reduces A . To this end we use the continuous map f defined by

$$f(K, L) = \zeta^K(1 + L)\omega.$$

If $K \notin \mathbf{WO}$ then by Proposition 3.9 $\mathbb{Q} \leq_s \zeta^K$. Then $f(K, L)$ has no initial or final segment which is scattered and by Proposition 2.2 we have $f(K, L) \in \mathbf{StS}$. If $K \in \mathbf{WO}$ then ζ^K is isomorphic to an ordinal power of ζ , which is scattered and strongly surjective by Proposition 3.6. In that case, using both Proposition 2.16 and Corollary 2.14, $f(K, L) \in \mathbf{StS}$ if and only if $(1 + L)\omega \in \mathbf{StS}$. Finally, since $(1 + L)\omega$ has a minimum, Proposition 2.5(1) and Theorem 1.2 tell us that $(1 + L)\omega \in \mathbf{StS}$ if and only if $L \in \mathbf{WO}$, which concludes. \square

4. BOUNDING THE COMPLEXITY OF \mathbf{StS} FROM ABOVE

Given a set $A \neq \emptyset$, the spaces $A^{\mathbb{N}}$ and $A^{\mathbb{N} \times \mathbb{N}}$ are endowed with the product topology of the discrete topology on A .

Given K, L in \mathbf{Lin} , $\mathbf{Emb}(L, K)$ is a closed subspace of K^L , so it is closed in $(\mathbb{N} \cup \{*\})^{\mathbb{N}}$ as well, where $*$ is a new symbol (we map the elements of $\mathbb{N} \setminus L$ to $*$). Similarly, the space $\mathbf{Epi}(L, K)$ is Π_2^0 in L^K and hence a Π_2^0 subspace of $(\mathbb{N} \cup \{*\})^{\mathbb{N}}$. Therefore both $\mathbf{Emb}(L, K)$ and $\mathbf{Epi}(L, K)$ are Polish spaces.

4.1. Some effective facts. We assume some familiarity with basic recursion theory. For effective descriptive set theory, we refer the reader to [Mos09, Section 3E] or to [Lo]. Notice that \mathbf{Lin} is a Polish recursive space in the sense of [Lo, Section 2.4.3].

We make a heavy use of [Mos09, Chapter 4]. Let us recall the following well-known facts ([Mos09, 4D.3] and [Lo, Section 5.1.5], respectively). We state them in relativized form, fixing a parameter K .

Fact 4.1. *If X and Y are recursive spaces and $A \subseteq X \times Y$ is $\Pi_1^1(K)$ then $\{x \in X \mid \exists y \in \Delta_1^1(x, K)(x, y) \in A\}$ is $\Pi_1^1(K)$.*

Fact 4.2. *Let X be a recursive space, Y a $\Pi_1^1(K)$ subset of a recursive space, and $A \subseteq X \times Y$ a $\Pi_1^1(K)$ set. Then there exists a $\Delta_1^1(K)$ function $f : \exists^\Delta A \rightarrow Y$ which uniformizes A on $\exists^\Delta A$, where*

$$x \in \exists^\Delta A \iff \exists y \in \Delta_1^1(x, K)((x, y) \in A).$$

Some basic operations on linear orders are effective.

Fact 4.3. *The operation $\star : L \mapsto L^\star$ is recursive.*

We now spell out what we mean by saying that the definition by pieces of Lemma 2.8 is Δ_1^1 .

Fact 4.4. *The following sets are Δ_1^1 :*

- (1) *for I an interval of \mathbb{Z} , the set of $(K, (K_i)_{i \in I})$ such that $K \in \text{Lin}$ and $(K_i)_i$ is a nice family of K ,*
- (2) *the same with nice covering,*
- (3) *the set of triples (K, L, M) such that L and M are adjacent convex subsets of K ,*
- (4) *the set of triples (K, L, M) such that L and M are connected convex subsets of K ,*
- (5) *the set of pairs (L, n) such that n is the maximum of L .*

Notation.

- For I an interval of \mathbb{Z} , \mathbf{Pieces}_I stands for the set of all $(\bar{K}, \bar{L}) := ((K, (K_i)_{i \in I}), (L, (L_i)_{i \in I}))$ in $(\text{Lin} \times \text{Lin}^I)^2$ such that: $(K_i)_{i \in I}$ is a nice family of convex subsets of K ; $(L_i)_{i \in I}$ is a nice covering of L ; for any $i \in I$ when K_i and K_{i+1} are not adjacent then L_i has a maximum or L_{i+1} has a minimum; and if K_i and K_{i+1} are connected, so are L_i and L_{i+1} .
- Call Epi the space $(\mathbb{N} \cup \{*\})^{\mathbb{N}} \cup \{\perp\}$ where \perp is not an element of $(\mathbb{N} \cup \{*\})^{\mathbb{N}}$; Epi is equipped with the smallest Polish topology extending that of $(\mathbb{N} \cup \{*\})^{\mathbb{N}}$ and making $\{\perp\}$ a clopen set.

Fact 4.5. *For any interval I of \mathbb{Z} , \mathbf{Pieces}_I is Δ_1^1 and so is the map*

$$\mathbf{DefPieces}_I : \mathbf{Pieces}_I \times \text{Epi}^I \longrightarrow \text{Epi}$$

$$((\bar{K}, \bar{L}), (\sigma_i)_{i \in I}) \longmapsto \begin{cases} \sigma & \text{if } \forall i \in I \ \sigma_i \in \text{Epi}(L_i, K_i) \\ \perp & \text{otherwise,} \end{cases}$$

where $\sigma \in \text{Epi}(L, K)$ is given by Lemma 2.8.

The explicit dependence on I will be omitted, and we shall write simply $\mathbf{DefPieces}$ to denote this function.

Fact 4.6. *If $K \in \text{Scat}$ then any convex suborder of K is $\Delta_1^1(K)$.*

Proof. Fix $K \in \text{Scat}$. First notice that K has countably many convex subsets (see [R82, Exercise 5.33.1]). Moreover the set of convex suborders of K is a $\Pi_1^0(K)$, and hence $\Sigma_1^1(K)$, subset of $2^{\mathbb{N} \times \mathbb{N}}$. So an application of Harrison's Effective Perfect Set Theorem ([Mos09, Theorem 4F.1]) allows to conclude. \square

Definition 4.7. Given A and B subsets of Lin , denote by $A + B$ the set $\{K \in \text{Lin} \mid \exists K_0 (K_0 \text{ is an initial segment of } K \wedge K_0 \in A \wedge K \setminus K_0 \in B)\}$.

Define then $n \cdot A$ for $n \geq 1$ by induction on n by $1 \cdot A = A$ and $(n+1) \cdot A = n \cdot A + A$, finally $\text{FinSum}(A)$ stands for $\bigcup_{n \in \mathbb{N}} (n+1) \cdot A$.

Fact 4.8. *If \mathbf{A} and \mathbf{B} are two $\Pi_1^1(K)$ subsets of \mathbf{Scat} , then so are $\mathbf{A} + \mathbf{B}$ and $\mathbf{FinSum}(\mathbf{A})$.*

Proof. The class $\Pi_1^1(K)$ is closed by effective countable unions, so it suffices to prove the statement for $\mathbf{A} + \mathbf{B}$. This comes from Facts 4.6 and 4.1. \square

4.2. Uniformizations for epimorphisms.

Definition 4.9. Given $K \in \mathbf{Lin}$ we say that K admits a Δ_1^1 -uniformization if there exists a $\Delta_1^1(K)$ map $\Phi_K : \mathbf{Lin} \rightarrow \mathbf{Epi}$ such that $\Phi_K(L) \in \mathbf{Epi}(L, K)$ when $L \leq_s K$ and $\Phi_K(L) = \perp$ when $L \not\leq_s K$.

Given $\mathbf{A} \subseteq \mathbf{Lin}$, if all $K \in \mathbf{A}$ admit a Δ_1^1 -uniformization we say that \mathbf{A} has the Δ_1^1 -uniformization for epimorphisms.

Our goal is to show in Theorem 4.16 below that \mathbf{Scat} has the Δ_1^1 -uniformization for epimorphisms.

Fact 4.10. *If L admits a Δ_1^1 -uniformization, so does L^* .*

Proof. Define $\Phi_{L^*}(M) = \Phi_L(M^*)$. \square

Notice that a Δ_1^1 -uniformization of some $K \in \mathbf{Lin}$ is a Δ_1^1 subset of $\mathcal{U} = \mathbf{Lin} \times \mathbf{Epi}$. Following [Lo, Section 5.1.1], we call $(\mathbf{D}^1, \mathbf{W}^1)$ the coding of Δ_1^1 -subsets of \mathcal{U} . Recall that we have

- \mathbf{D}^1 is a Π_1^1 subset of $\mathbb{N}^{\mathbb{N}}$, \mathbf{W}^1 is a Δ_1^1 subset of $\mathbf{D}^1 \times \mathcal{U}$
- $\{\mathbf{W}_\alpha^1 \mid \alpha \in \mathbf{D}^1\}$ is the set of Δ_1^1 subsets of \mathcal{U} and for any K , $\{\mathbf{W}_\alpha^1 \mid \alpha \in \mathbf{D}^1 \text{ and } \alpha \text{ recursive in } K\}$ is the set of $\Delta_1^1(K)$ subsets of \mathcal{U} .

Fact 4.11. *There is a partial Δ_1^1 map: $\mathbf{Lin} \rightarrow \mathbf{D}^1$, $K \mapsto \alpha_K$ such that for every $K \in \mathbf{Lin}$ admitting a Δ_1^1 -uniformization, $\alpha_K \in \mathbf{D}^1$ is a code of a Δ_1^1 -uniformization $\Phi_K = \mathbf{W}_{\alpha_K}^1$.*

Proof. The relation

α is the code of a function that uniformizes K

is a Π_1^1 subset of $\mathbf{D}^1 \times \mathbf{Lin}$ by definition:

α codes a uniformization of $K \iff \alpha$ codes a function Φ and

$$\forall L \in \mathbf{Lin} \left((L \not\leq_s K \wedge \Phi(L) = \perp) \vee \Phi(L) \in \mathbf{Epi}(L, K) \right).$$

The result then follows using Fact 4.2. \square

Proposition 4.12. *If $\mathbf{A}, \mathbf{B} \subseteq \mathbf{Scat}$ have the Δ_1^1 -uniformization for epimorphisms, then so does $\mathbf{A} + \mathbf{B}$.*

In particular, $\mathbf{FinSum}(\mathbf{A})$ has the Δ_1^1 -uniformization for epimorphisms.

Proof. We fix $K \in \mathbf{A} + \mathbf{B}$ and $K_0 \in \mathbf{A}$, $K_1 \in \mathbf{B}$ such that K_0 is an initial segment of K and $K_1 = K \setminus K_0$. Since \mathbf{A} and \mathbf{B} have the Δ_1^1 -uniformization for epimorphisms, for $i = 0, 1$ there exists a Δ_1^1 -uniformization Φ_{K_i} for K_i . As K_i is a convex subset of $K \in \mathbf{Scat}$, by Fact 4.6 $\Delta_1^1(K_i)$

is included in $\Delta_1^1(K)$. Recalling that each Φ_{K_i} is a $\Delta_1^1(K_i)$ map, this implies that Φ_{K_i} is a $\Delta_1^1(K)$ map.

By Lemma 2.8 (and using the fact that K_0 and K_1 are adjacent but not connected), L satisfies $L \leq_s K$ if and only if there is a nice covering (L_0, L_1) of L satisfying $L_0 \leq_s K_0$ and $L_1 \leq_s K_1$. Since Φ_{K_i} allows to check whether $L_i \leq_s K_i$ in a $\Delta_1^1(K)$ way and using Fact 4.4(2), the set \mathbb{C} of triples (L, L_0, L_1) satisfying the latter is $\Delta_1^1(K)$.

We now use the effective version of Lusin-Novikov's "small section" uniformization result (see [Mos09, 4F.6]: the statement there is not effective, but the hint proves the effective version) to obtain two $\Delta_1^1(K)$ functions Ψ_0 and Ψ_1 with domain $\{L \mid \exists L_0, L_1 (L, L_0, L_1) \in \mathbb{C}\}$ such that $(L, \Psi_0(L), \Psi_1(L)) \in \mathbb{C}$ holds for any $L \leq_s K$.

We can now define $\Phi_K : \text{Lin} \rightarrow \text{Epi}$ by setting $\Phi_K(L)$ to be the map defined by pieces from $\Phi_{K_0} \circ \Psi_0(L)$ and $\Phi_{K_1} \circ \Psi_1(L)$ when $\Psi_0(L)$ and $\Psi_1(L)$ are defined, and $\Phi_K(L) = \perp$ otherwise. By Fact 4.5, Φ_K is a $\Delta_1^1(K)$ map and in fact a Δ_1^1 -uniformization of K . \square

We recall (some version of) Hausdorff's hierarchy of countable scattered linear orders.

- Call $\text{Scat}_0 \subseteq \text{Lin}$ the class of singleton orders,
- an element of Lin is in Scat_α when it is isomorphic to a finite sum, an ω -sum or an ω^* -sum of elements of $\bigcup_{\beta < \alpha} \text{Scat}_\beta$.

Hausdorff proved that $\text{Scat} = \bigcup_{\alpha < \omega_1} \text{Scat}_\alpha$, so for $K \in \text{Scat}$ we define the *Hausdorff rank* of K :

$$\text{rk}_H(K) = \min\{\alpha < \omega_1 \mid K \in \text{Scat}_\alpha\}.$$

We have $\text{Scat}_\alpha = \{K \in \text{Scat} \mid \text{rk}_H(K) \leq \alpha\}$ and we can set $\text{Scat}_{<\alpha} = \bigcup_{\beta < \alpha} \text{Scat}_\beta = \{K \in \text{Scat} \mid \text{rk}_H(K) < \alpha\}$.

Recall that if L is a suborder of K , then $\text{rk}_H(L) \leq \text{rk}_H(K)$ holds, and that rk_H is a Π_1^1 -norm (see [Mos09, Section 4B]). In particular, Scat_α and $\text{Scat}_{<\alpha}$ are $\Delta_1^1(\alpha)$. Moreover, for any $K \in \text{Scat}$, using [Mos09, Theorem 4D.1(iii)] we have $\text{rk}_H(K) \in \Delta_1^1(K)$.

To prove that if Scat_α has the Δ_1^1 -uniformization for epimorphism so does $\text{Scat}_{\alpha+1}$ we only need to handle the case of ω -sums (as for the case of ω^* -sums we can use Fact 4.10, and the case of finite sums is handled by Proposition 4.12). We use the following notion, implicitly used in [La79].

Definition 4.13. We say that an order K is *stable* if and only if for all $k \in K$ we have $K \equiv_s [k, \rightarrow)_K$.

Since it is always the case that $[k, \rightarrow)_K \leq_s K$, K stable really means $K \leq_s [k, \rightarrow)_K$ for all $k \in K$. Notice that in particular a stable K has a minimum and that the only stable orders with a maximum are the singletons.

Recall from [La79, CCM15] that the class of countable linear orders is well-quasi-ordered (wqo) under epimorphisms. In particular, (\mathbf{Lin}, \leq_s) is well-founded. We use the following observation, due to Landraitis [La79, Lemma 2.2].

Lemma 4.14. *Every $K \in \mathbf{Lin}$ has a stable final segment.*

Proof. Fix $K \in \mathbf{Lin}$ and look for $a \in K$ such that $[a, \rightarrow)_K$ is stable. Fix a sequence $(a_n)_{n \in \mathbb{N}}$ monotone and cofinal in K . The sequence $([a_n, \rightarrow)_K)_{n \in \mathbb{N}}$ is \leq_s -decreasing so there is $N \in \mathbb{N}$ such that $([a_n, \rightarrow)_K)_{n \geq N}$ is \equiv_s -constant, for \leq_s is well-founded on \mathbf{Lin} . Choose then $a = a_N$. For any $k \geq_K a$ there is $n > N$ such that $k \leq_K a_n$ holds, and finally

$$[a, \rightarrow)_K \leq_s [a_n, \rightarrow)_K \leq_s [k, \rightarrow)_K,$$

so $[a, \rightarrow)_K$ is a stable final segment of K . \square

We need the following characterization of stability (also essentially contained in [La79, Lemma 2.2]).

Fact 4.15. *An admissible linear order K is stable if and only if it has a minimum and for all $a_0 \leq_K a_1$ and a in K there are b_0, b_1 in K such that $a \leq_K b_0 \leq_K b_1$ and $[a_0, a_1]_K \leq_s [b_0, b_1]_K$.*

Proof. It is immediate that if K is stable then it has the desired property, so it is enough to show the converse. Given any $k \in K$, take $(a_i)_{i \in \mathbb{N}}$ a cofinal monotone sequence in K , with $a_0 = \min K$ and $a_1 = k$. Use the hypothesis to find a sequence $(b_i^0, b_i^1)_{i \in \mathbb{N}}$ such that we have

- $a_{i+1} \leq_K b_i^0 \leq_K b_i^1 \leq_K b_{i+1}^0$,
- $[a_i, a_{i+1}]_K \leq_s [b_i^0, b_i^1]_K$.

Notice that since $[b_0^0, b_0^1]_K \leq_s [k, b_0^1]_K$ we can assume that $b_0^0 = k$.

We can apply Lemma 2.8 to define by pieces a surjection showing that $K \leq_s [k, \rightarrow)_K$. \square

Theorem 4.16. *\mathbf{Scat} has the Δ_1^1 -uniformization for epimorphisms.*

Proof. We prove inductively on α that \mathbf{Scat}_α has the Δ_1^1 -uniformization for epimorphisms. Take $K \in \mathbf{Scat}_0$, so that there is $n \in \mathbb{N}$ such that $K = \{n\}$. For $m \in \mathbb{N}$ define $\tau_{n,m} : \mathbb{N} \rightarrow \mathbb{N} \cup \{*\}$ by letting $\tau_{n,m}(n) = m$, and $\tau_{n,m}(i) = *$ if $i \neq n$; then

$$\Phi_K : \mathbf{Lin} \longrightarrow \mathbf{Epi}$$

$$L \longmapsto \begin{cases} \tau_{n,m} & \text{if } L = \{m\} \text{ for some } m \\ \perp & \text{otherwise} \end{cases}$$

is a Δ_1^1 -uniformization¹ of K .

¹Notice that this Φ_K is not continuous!

Fix now $\alpha < \omega_1$ with $\alpha \geq 1$, and suppose that $\mathbf{Scat}_{<\alpha} = \bigcup_{\beta < \alpha} \mathbf{Scat}_\beta$ has the Δ_1^1 -uniformization for epimorphisms. We need to prove that \mathbf{Scat}_α has it too.

By Proposition 4.12, $\mathbf{FinSum}(\mathbf{Scat}_{<\alpha})$ has the Δ_1^1 -uniformization for epimorphisms. Recall also that, by Fact 4.11, there is a Δ_1^1 map that for any $L \in \mathbf{FinSum}(\mathbf{Scat}_{<\alpha})$ chooses the code α_L of a Δ_1^1 -uniformization Φ_L for L .

Calling \mathbf{A} the set of stable elements of \mathbf{Scat}_α , Lemma 4.14 yields that \mathbf{Scat}_α can be defined as

$$\mathbf{FinSum}(\mathbf{Scat}_{<\alpha}) \cup (\mathbf{FinSum}(\mathbf{Scat}_{<\alpha}) + \mathbf{A}) \cup \star(\mathbf{FinSum}(\mathbf{Scat}_{<\alpha}) + \mathbf{A}),$$

where we are using obvious notations.

Fact 4.10 and Proposition 4.12 tell us that if \mathbf{A} has the Δ_1^1 -uniformization for epimorphisms, then so does \mathbf{Scat}_α .

Fix $K \in \mathbf{A}$ and let k_0 be the minimum of K . If K has a maximum, it is a singleton and we already know that it has a Δ_1^1 -uniformization. Thus we can assume that K is a stable ω -sum of elements of $\mathbf{Scat}_{<\alpha}$. To define a $\Delta_1^1(K)$ -uniformization Φ_K of K we use the Δ_1^1 -uniformizations $\Phi_{[k,k']_K}$ of $[k,k']_K$. Notice that the $\Phi_{[k,k']_K}$ are $\Delta_1^1(K)$ as well, because intervals are $\Delta_1^0(K)$.

We now define $\Phi_K(L)$ distinguishing three cases, each of them defined by a Δ_1^1 property.

If L has no minimum (a Π_2^0 condition) then notice that $L \not\leq_s K$ and let $\Phi_K(L) = \perp$.

If L has two extrema (a Σ_2^0 condition) then notice that $L \leq_s K$ if and only if there is $k_1 \in K$ such that $L \leq_s [k_0, k_1]_K$ holds, if and only if $\exists k_1 \in K \Phi_{[k_0, k_1]_K}(L) \neq \perp$. In case k_1 does exist, choose k_1 minimal (as a natural number) and define $\Phi_K(L)$ by pieces from $\Phi_{[k_0, k_1]_K}(L)$ using the function **DefPieces** of Fact 4.5. Otherwise let $\Phi_K(L) = \perp$.

The last case is when L has minimum but no maximum (a Δ_3^0 condition). Let $\mathbf{B} = \{L \in \mathbf{Lin} \mid L \text{ has minimum but no maximum}\}$. If $L \in \mathbf{B}$ we denote by $\{l_i \mid i \in \mathbb{N}\}$ the canonical cofinite sequence in L , defined by letting l_0 be the minimum of L , and l_{i+1} be the least (as natural number) $l \in L$ such that $l_i <_L l$. Notice that the map $L \mapsto \{l_i \mid i \in \mathbb{N}\}$ is Δ_1^1 on the Δ_3^0 set \mathbf{B} .

Claim. *There exists a $\Delta_1^1(K)$ function $f : \mathbf{B} \rightarrow K^\mathbb{N}$ such that, writing $f(L)(i) = k_i^L$, we have that the sequence $\{k_i^L \mid i \in \mathbb{N}\}$ is strictly increasing and cofinal in K and moreover $L \leq_s K$ if and only if $[l_i, l_{i+1}]_L \leq_s [k_i^L, k_{i+1}^L]_K$ for every $i \in \mathbb{N}$.*

Proof of claim. Given $L \in \mathbf{B}$ we uniformly define in a $\Delta_1^1(K)$ way the sequence $\{k_i^L \mid i \in \mathbb{N}\}$ and an auxiliary sequence $\{m_i^L \mid i \in \mathbb{N}\} \in 2^\mathbb{N}$ by induction on i . The intuition for m_i^L is that as long as $m_i^L = 0$ we are still hoping to show that $L \leq_s K$, while when we set $m_i^L = 1$

we actually know that $L \not\leq_s K$ and we just need to make sure that $\{k_i^L \mid i \in \mathbb{N}\}$ is cofinal in K .

As $K \in \mathbf{B}$, $\{k_i \mid i \in \mathbb{N}\}$ stands for the canonical cofinite sequence in K . First, $k_0^L = k_0$ is the minimum of K and $m_0^L = 0$. Assuming we have already defined k_i^L and m_i^L we proceed as follows. If $m_i^L = 0$ we look for $k \in K$ such that $[l_i, l_{i+1}]_L \leq_s [k_0^L, k]_K$. If we succeed, we let k be the least (as natural number) such k and, using the stability of K and Fact 4.15, find the least (code for) a pair $(\hat{k}, k_{i+1}^L) \in K^2$ such that $[k_0^L, k]_K \leq_s [\hat{k}, k_{i+1}^L]_K$ and $\max_K(k_i^L, k_i) \leq_K \hat{k}$. This way we defined k_{i+1}^L , and we set also $m_{i+1}^L = 0$. Notice that in this case we have

$$[l_i, l_{i+1}]_L \leq_s [k_0^L, k]_K \leq_s [\hat{k}, k_{i+1}^L]_K \leq_s [k_i^L, k_{i+1}^L]_K.$$

If either the search for $k \in K$ such that $[l_i, l_{i+1}]_L \leq_s [k_0^L, k]_K$ fails or $m_i^L = 1$, we let $m_{i+1}^L = 1$ and k_{i+1}^L be the least (as natural number) $k \in K$ such that $\max_K(k_i^L, k_i) <_K k$.

Since we made sure that $k_i \leq_K k_i^L$ the sequence $\{k_i^L \mid i \in \mathbb{N}\}$ is indeed cofinal in K .

Now notice that if $m_i^L = 0$ for every i then $[l_i, l_{i+1}]_L \leq_s [k_i^L, k_{i+1}^L]_K$ for every i ; using a definition by pieces we find a witness to $L \leq_s K$. If instead $m_i^L = 1$ for some i let i be the least such. Then $[l_i, l_{i+1}]_L \not\leq_s [k_i^L, k_{i+1}^L]_K$ and there is no $k \in K$ such that $[l_i, l_{i+1}]_L \leq_s [k_0^L, k]_K$. The latter fact implies $L \not\leq_s K$. \square

Now, using the claim, we can define Φ_K on \mathbf{B} . If $[l_i, l_{i+1}]_L \leq_s [k_i^L, k_{i+1}^L]_K$ for every i (a $\Delta_1^1(K)$ condition) then $\Phi_K(L)$ can be defined applying the function **DefPieces** from Fact 4.5 to the epimorphisms $\Phi_{[k_i^L, k_{i+1}^L]_K}([l_i, l_{i+1}]_L)$. If instead $[l_i, l_{i+1}]_L \not\leq_s [k_i^L, k_{i+1}^L]_K$ for some i we set $\Phi_K(L) = \perp$. \square

We can finally pinpoint the complexity of **StS**, but first the complexity of $\mathbf{StS} \cap \mathbf{Scat}$.

Corollary 4.17. *The set of scattered strongly surjective orders is Π_1^1 and Π_1^1 -complete.*

Proof. The fact that $\mathbf{StS} \cap \mathbf{Scat}$ is Π_1^1 -hard is contained in Proposition 3.2.

Given an order K say that a Δ_1^1 -uniformization Φ of K is strong if for all L such that $L \leq_i K$ we have $\Phi(L) \neq \perp$. By Theorem 4.16 K is scattered and strongly surjective if and only if it is scattered and admits a strong Δ_1^1 -uniformization.

This gives in turn, using Fact 4.1, a Π_1^1 definition of $\mathbf{StS} \cap \mathbf{Scat}$. \square

Corollary 4.18. *The set **StS** is the union of a Σ_1^1 set and a Π_1^1 set. It is in particular $\check{D}_2(\Pi_1^1)$, and in fact $\check{D}_2(\Pi_1^1)$ -complete.*

Proof. By Proposition 3.3, Corollary 4.17 and Theorem 3.11. \square

5. LOOKING FOR UNCOUNTABLE STRONGLY SURJECTIVE ORDERS

5.1. Classical examples are not strongly surjective. Recall that Proposition 2.5(4) states that a strongly surjective linear order can have at most the cardinality of the continuum. Here we show that the most common orders of size the continuum and those that can be obtained from them using basic operations are not strongly surjective. We use different techniques, and for some linear orders we have different proofs that they are not strongly surjective.

We first give a cardinality obstruction for strong surjectivity of sub-orders of \mathbb{R} .

Theorem 5.1. *Let $\aleph_0 < \kappa \leq 2^{\aleph_0}$, and assume $2^{\aleph_0} < 2^\kappa$. Then no $X \subseteq \mathbb{R}$ of cardinality κ can be strongly surjective.*

Proof. We use a counting argument: there are more subsets of X than order-preserving maps from X to X .

Since $X \subseteq \mathbb{R}$ we can find a countable subset D of X such that for all $x, x' \in X$, if $x < x'$ holds then there is a $d \in D$ with $x \leq d \leq x'$, and such that moreover the endpoints of X belong to D , if they exist. Every order-preserving map from X to X is the extension of an order-preserving map from D to X , and there are at most continuum many of those:

$$|\{f : D \rightarrow X \mid f \text{ order-preserving}\}| \leq |X^D| = \kappa^{\aleph_0} = 2^{\aleph_0}.$$

Fix now $f : D \rightarrow X$ order-preserving, and compute how many order-preserving extensions $g : X \rightarrow X$ of f there can be. Take $x \in X \setminus D$, we can pick for $g(x)$ any $y \in X$ that satisfies

$$(2) \quad \forall d \in (\leftarrow, x)_X \cap D \quad \forall d' \in (x, \rightarrow)_X \cap D \quad f(d) \leq y \leq f(d').$$

Call I_x the convex set of all points y satisfying (2). If I_x is trivial, that is empty or reduced to a singleton, then there is at most one order-preserving extension of f to x . Notice now that by the properties of D , for $x \neq x'$ in $X \setminus D$ the sets I_x and $I_{x'}$ are disjoint. Since $X \subseteq \mathbb{R}$ there can be only countably many non-trivial I_x , and each of these yields at most κ possible extensions, so for a fixed f we have

$$|\{g : X \rightarrow X \mid g \text{ order-preserving and } f \subset g\}| \leq \kappa^{\aleph_0} = 2^{\aleph_0}.$$

All in all we have at most continuum many order-preserving maps from X to X , but X has even more subsets by hypothesis so it cannot be strongly surjective. \square

Corollary 5.2. *\mathbb{R} and $\mathbb{R} \setminus \mathbb{Q}$ are not strongly surjective.*

Theorem 5.1 and its proof do not provide a concrete L such that $L \leq_i \mathbb{R}$ yet $L \not\leq_s \mathbb{R}$ (and similarly for $\mathbb{R} \setminus \mathbb{Q}$). A useful technique to prove that a linear order does not admit epimorphisms onto another one is to compare their gaps. Recall that a *gap* of K is given by a non-empty initial segment $A \subseteq K$ with no maximum such that $K \setminus A$

is non-empty and has no minimum. Let $G(K)$ be the set of gaps of K linearly ordered by \subseteq .

Proposition 5.3. *If L and K are linear orders such that $L \leq_s K$ then $G(L) \leq_i G(K)$.*

Proof. If f is an epimorphism from K onto L then $A \mapsto f^{-1}(A)$ is an injection from $G(L)$ to $G(K)$. \square

Corollary 5.4. *Any linear order L with $|G(L)| < 2^{\aleph_0}$ is such that $\mathbb{Q} \not\leq_s L$.*

Hence \mathbb{Q} witnesses the fact that $\mathbb{R}, \mathbb{R} \setminus \mathbb{Q}, \mathbb{Z}^{\mathbb{N}}$, and, for every countable $\alpha \geq \omega$, 2^α ordered lexicographically are not strongly surjective.

Proof. The first part follows immediately from the Proposition because $|G(\mathbb{Q})| = 2^{\aleph_0}$.

Each of the linear orders considered in the second part of the statement (and indeed each short uncountable linear order) is non-scattered. Now observe that \mathbb{R} and 2^α are complete (that is, they have no gaps), while $|G(\mathbb{R} \setminus \mathbb{Q})| = |G(\mathbb{Z}^{\mathbb{N}})| = \aleph_0$. The statements about \mathbb{R} and $\mathbb{R} \setminus \mathbb{Q}$ are obvious.

To see that 2^α is complete let A be a non-empty subset of 2^α . Define inductively $x \in 2^\alpha$ as follows: given $\beta \in \alpha$ and assuming that $x(\gamma)$ has been constructed for every $\gamma \in \beta$, let $x(\beta) \in \{0, 1\}$ be the least value such that $x|_{\beta+1}$ majorizes $\{z|_{\beta+1} \mid z \in A\}$. Then x majorizes A . Moreover, if y majorizes A and $y <_{lex} x$, let $s = y \cap x$; then $s \not\supseteq y, s \not\supseteq x$, contrary to the definition of x . So $x = \sup A$.

To see that $\mathbb{Z}^{\mathbb{N}}$ has only countably many gaps let $L = \mathbb{Z}^{\mathbb{N}} \cup (\mathbb{Z}^{<\mathbb{N}} \setminus \{\emptyset\})$ where the order on $\mathbb{Z}^{\mathbb{N}}$ is extended to L by ordering $\mathbb{Z}^{<\mathbb{N}} \setminus \{\emptyset\}$ lexicographically and letting $x <_L s$ if and only if $x|_{\text{lh}(s)} \leq_{lex} s$ for every $x \in \mathbb{Z}^{\mathbb{N}}$ and $s \in \mathbb{Z}^{<\mathbb{N}} \setminus \{\emptyset\}$. Then L is complete. \square

Notice that the fact that 2^{\aleph_0} and $\mathbb{Z}^{\mathbb{N}}$ are not strongly surjective shows that Corollary 2.14, stating that strongly surjective orders are closed under finite products, cannot be extended to infinite products.

The next natural candidates for being uncountable strongly surjective orders are the finite products obtained by using $\mathbb{R} \setminus \mathbb{Q}, \mathbb{R}$ and possibly some countable orders as factors. We show however that no uncountable strongly surjective order can be obtained in this way.

Lemma 5.5. *Let K, L and M be linear orders. Suppose that $K \not\leq_s L$ and that $K' \not\leq_s M$ for any convex subset K' of K that has more than one point. Then we have $K \not\leq_s ML$.*

Proof. Suppose we have $\varphi \in \text{Epi}(K, ML)$ and consider $f : L \rightarrow \mathcal{P}(K)$ defined by $f(r) = \varphi(M \times \{r\})$. There is some $r \in L$ such that $f(r)$ is not a singleton, otherwise f would induce an epimorphism from L onto K . But then $f(r) \leq_s M$ and $f(r)$ is a convex subset of K with more than one point, which is again impossible. \square

Definition 5.6. If κ is an infinite cardinal, a linear order L is κ -dense if it has no end points and between any two distinct elements of L there are exactly κ elements of L .

Lemma 5.7. *There is a 2^{\aleph_0} -dense suborder M of $\mathbb{R} \setminus \mathbb{Q}$ such that every interval $[x, y]_M$, for $x <_M y$, has 2^{\aleph_0} gaps.*

Proof. Let $\{I_n\}_{n \in \mathbb{N}}$ be the set of the traces on $\mathbb{R} \setminus \mathbb{Q}$ of the real intervals with rational endpoints. In each I_n we define subsets $X_n = \{x_\beta^n\}_{\beta \in 2^{\aleph_0}}$ and $Y_\beta^n = \{y_{\beta m}^n\}_{m \in \mathbb{N}}$, for $\beta \in 2^{\aleph_0}$, such that the elements $x_\beta^n, y_{\beta m}^n$ are all distinct. Fix $\beta < 2^{\aleph_0}$ and $n \in \mathbb{N}$. Suppose that $x_{\beta'}^n$ and $y_{\beta' m}^n$ are defined for all $\beta' < \beta$ and $n', m \in \mathbb{N}$, as well as $x_\beta^{n'}$ and $y_{\beta m}^{n'}$ for $n' < n$ and $m \in \mathbb{N}$. Notice that the set A of these elements, if non-empty, has cardinality $\max(\aleph_0, |\beta|) < 2^{\aleph_0}$. So pick any $x_\beta^n \in I_n \setminus A$ and distinct $y_{\beta m}^n \in I_n \setminus (A \cup \{x_\beta^n\})$ such that

$$\sup\{y_{\beta m}^n \mid m \in \mathbb{N}, y_{\beta m}^n < x_\beta^n\} = x_\beta^n = \inf\{y_{\beta m}^n \mid m \in \mathbb{N}, x_\beta^n < y_{\beta m}^n\}.$$

It follows that every x_β^n is a gap in $M = (\mathbb{R} \setminus \mathbb{Q}) \setminus \{x_\beta^n \mid \beta < 2^{\aleph_0}, n \in \mathbb{N}\}$. Moreover, if $x, y \in M, x < y$, then $[x, y]_M$ contains some $I_n \cap M$, which has the cardinality of the continuum (containing all $y_{\beta m}^n$) and has continuum many gaps (at least all x_β^n). \square

Theorem 5.8. *Let $L = \prod_{0 \leq i \leq n} L_i$, where for each i either L_i is countable or $\mathbb{R} \setminus \mathbb{Q} \leq_i L_i$ and $|G(L_i)| < 2^{\aleph_0}$ ($\mathbb{R} \setminus \mathbb{Q}$ and \mathbb{R} are instances of such linear orders). If L is uncountable, then L is not strongly surjective.*

Proof. Suppose that L is uncountable, so at least one of the factors is uncountable. Let M be the order given by Lemma 5.7. Since $M \leq_i \mathbb{R} \setminus \mathbb{Q}$ we have $M \leq_i L$. It then suffices to show that $M \not\leq_s L$, and this can be done by induction on n .

For $n = 0$, notice that $M \not\leq_s L$ since $|G(L)| < 2^{\aleph_0} = |G(M)|$.

If the statement holds for n , let $L = \prod_{0 \leq i \leq n+1} L_i$. If $\prod_{1 \leq i \leq n+1} L_i$ is countable (by cardinality reasons), or by inductive hypothesis, $M \not\leq_s \prod_{1 \leq i \leq n+1} L_i$. Moreover, if M' is any convex subset of M containing more than one point, then $M' \not\leq_s L_0$ either by cardinality reasons (if L_0 is countable) or by the fact that M' has more gaps than L_0 (if L_0 is uncountable). Now apply Lemma 5.5. \square

We already argued that 2^{\aleph_0} and \mathbb{Z}^{\aleph_0} ordered lexicographically are not strongly surjective. However the gap method does not apply to other natural infinite lexicographic products, such as \mathbb{Q}^{\aleph_0} , \mathbb{R}^{\aleph_0} , and $(\mathbb{R} \setminus \mathbb{Q})^{\aleph_0}$, which have 2^{\aleph_0} many gaps. First we show that products such as \mathbb{R}^{\aleph_0} and $(\mathbb{R} \setminus \mathbb{Q})^{\aleph_0}$ are not strongly surjective.

Theorem 5.9. *For every $k \in \mathbb{N}$ let L_k be a linear order with at least two elements such that for every convex set $K \subseteq L_k$ we have $\mathbb{Q} \not\leq_s K$. Then $\mathbb{Q} \not\leq_s \prod_{k \in \mathbb{N}} L_k$, where the product is ordered lexicographically.*

Proof. First of all notice that the hypothesis on L_k implies that every convex subset of L_k does not surject onto $1 + \mathbb{Q}$, $\mathbb{Q} + 1$, and $1 + \mathbb{Q} + 1$ either.

Suppose $f : \prod_{k \in \mathbb{N}} L_k \rightarrow \mathbb{Q}$ is an epimorphism. Our goal is to define an embedding $g : 2^{\mathbb{N}} \rightarrow \prod_{k \in \mathbb{N}} L_k$ such that f is injective on the range of g , thus reaching a contradiction because \mathbb{Q} is countable.

If $s \in \prod_{i < k} L_i$ for some $k \in \mathbb{N}$ we let $N_s = \{z \in \prod_{k \in \mathbb{N}} L_k \mid s \subset z\}$. To define g we define $h : 2^{<\mathbb{N}} \rightarrow \bigcup_{k \in \mathbb{N}} \prod_{i < k} L_i$ such that $h(s) \in \prod_{i < 2k} L_i$ when $s \in 2^k$, $h(s) \subset h(t)$ when $s \subset t$, $f(N_{h(s)})$ is a convex subset of \mathbb{Q} with at least two elements, and $f(N_{h(s \smallfrown 0)}) \cap f(N_{h(s \smallfrown 1)}) = \emptyset$. Then we set $g(x) = \bigcup_{k \in \mathbb{N}} h(x \upharpoonright k)$, so that it is immediate that if $x, y \in 2^{\mathbb{N}}$ are distinct then $f(g(x)) \neq f(g(y))$.

The definition of $h(s)$ is by recursion on the length of s , starting from $h(\emptyset) = \emptyset$. Thus we assume that s has length k and $h(s)$ is defined respecting the conditions, so that $f(N_{h(s)})$ is isomorphic to one of \mathbb{Q} , $1 + \mathbb{Q}$, $\mathbb{Q} + 1$, and $1 + \mathbb{Q} + 1$. Consider the map that sends $\ell \in L_{2k}$ to $f(N_{h(s) \smallfrown \ell}) \subseteq \mathbb{Q}$: since $f(N_{h(s)}) \not\leq_s L_{2k}$, for some $\ell_0 \in L_{2k}$ we have that $f(N_{h(s) \smallfrown \ell_0})$ is not a singleton. Since L_{2k} has at least two elements ℓ_0 is either not the maximum or not the minimum of L_{2k} . Let us assume it is not the maximum (otherwise we reason symmetrically). Then $(\ell_0, \rightarrow)_{L_{2k}}$ is a nonempty convex subset of L_{2k} and hence does not surject onto $\bigcup_{\ell >_{L_{2k}} \ell_0} f(N_{h(s) \smallfrown \ell})$, which is isomorphic to one of \mathbb{Q} , $1 + \mathbb{Q}$, $\mathbb{Q} + 1$, and $1 + \mathbb{Q} + 1$. Therefore we can find $\ell_1 >_{L_{2k}} \ell_0$ such that $f(N_{h(s) \smallfrown \ell_1})$ is also not a singleton. Notice that it might be that $f(N_{h(s) \smallfrown \ell_1})$ intersect $f(N_{h(s) \smallfrown \ell_0})$ in a common endpoint. Thus we go to the next level and find $\ell_{00} <_{L_{2k+1}} \ell_{01} <_{L_{2k+1}} \ell_{10} <_{L_{2k+1}} \ell_{11}$ such that $f(N_{h(s) \smallfrown \ell_{ij}})$ is not a singleton for every i, j . Consequently $f(N_{h(s) \smallfrown \ell_{00}}) \cap f(N_{h(s) \smallfrown \ell_{11}}) = \emptyset$. Let $h(s \smallfrown i) = h(s) \smallfrown \ell_{ii}$ for $i = 0, 1$. \square

Corollary 5.10. \mathbb{Q} witnesses that $2^{\mathbb{N}}$, $\mathbb{Z}^{\mathbb{N}}$, $\mathbb{R}^{\mathbb{N}}$, and $(\mathbb{R} \setminus \mathbb{Q})^{\mathbb{N}}$ ordered lexicographically are not strongly surjective.

Proof. First notice that 2 , \mathbb{Z} , \mathbb{R} , and $\mathbb{R} \setminus \mathbb{Q}$ satisfy the condition imposed by Theorem 5.9 on the L_i 's. Then observe that each of $2^{\mathbb{N}}$, $\mathbb{Z}^{\mathbb{N}}$, $\mathbb{R}^{\mathbb{N}}$, and $(\mathbb{R} \setminus \mathbb{Q})^{\mathbb{N}}$ is non-scattered. \square

To show that $\mathbb{Q}^{\mathbb{N}}$ is not strongly surjective we must use a different approach (obviously $\mathbb{Q} \leq_s \mathbb{Q}^{\mathbb{N}}$): we exploit the definability of epimorphisms in certain settings.

Theorem 5.11. No uncountable Borel suborder of $2^{\mathbb{N}}$, with the lexicographic order, is strongly surjective.

Proof. First, notice that the usual product topology and the order topology on $2^{\mathbb{N}}$ coincide. Indeed, for any $s \in 2^{<\mathbb{N}}$, the basic open set $N_s = \{x \in 2^{\mathbb{N}} \mid s \subseteq x\}$ is open in the order topology:

- $N_\emptyset = 2^\mathbb{N}$
- if $s = s'01^h$ for some $h > 0$, then $N_s = (s'01^{h-1}01^\infty, s'10^\infty)_{2^\mathbb{N}}$; similarly if $s = s'10^h$ for some $h > 0$
- if $s = 0^h$ for some $h > 0$, then $N_s = (\leftarrow, 0^{h-1}10^\infty)_{2^\mathbb{N}}$; similarly if $s = 1^h$ for some $h > 0$

Conversely, given any $x \in 2^\mathbb{N}$, fix $y \in (x, \rightarrow)_{2^\mathbb{N}}$ and set $s = x \cap y$. Observe that $x \in N_{s0}$ and $y \in N_{s1}$, so $N_{s1} \subseteq (x, \rightarrow)_{2^\mathbb{N}}$, which implies that $(x, \rightarrow)_{2^\mathbb{N}}$ is open in Cantor space. Similarly one proves that $(\leftarrow, x)_{2^\mathbb{N}}$ is open.

Now let $X \subseteq 2^\mathbb{N}$, and fix any order-preserving function $f : X \rightarrow 2^\mathbb{N}$. For any $s \in 2^{<\mathbb{N}}$ the subset $f^{-1}(N_s)$ of X is convex, so there exists a convex subset A of $2^\mathbb{N}$ such that $f^{-1}(N_s) = A \cap X$. By the above, if $x \in A$ and x is not an end point of A , then x is in the topological interior of A . This implies that A is the union of an open set plus at most two points (its end points, if they exist), so A is Borel in $2^\mathbb{N}$ and $f^{-1}(N_s)$ is Borel in X ; consequently, f is Borel.

If X is Borel in $2^\mathbb{N}$, then $f(X)$ must be analytic; so if X is uncountable, then X is not strongly surjective, since there exist non-analytic subsets of X onto which there can be no epimorphism. \square

Corollary 5.12. $2^\mathbb{N}$, $\mathbb{Z}^\mathbb{N}$ and $\mathbb{Q}^\mathbb{N}$ ordered lexicographically are not strongly surjective.

Proof. It is enough to show that $\mathbb{Z}^\mathbb{N}$ and $\mathbb{Q}^\mathbb{N}$, endowed with the product topology on the discrete topology, Borel embed order preservingly in $2^\mathbb{N}$ ordered lexicographically. Since the natural inclusion is a Borel order preserving embedding of $\mathbb{Z}^\mathbb{N}$ into $\mathbb{Q}^\mathbb{N}$, by the main theorem of [Lo89] it is enough to prove that $2^{\omega+1} \not\leq_i \mathbb{Q}^\mathbb{N}$. Notice that $2^{\omega+1}$ has uncountably many pairs of consecutive points, so if $2^{\omega+1} \leq_i \mathbb{Q}^\mathbb{N}$ then $\mathbb{Q}^\mathbb{N}$, being dense, should have uncountably many pairwise disjoint open intervals. However this is not the case, as every \leq_{lex} -open interval in $\mathbb{Q}^\mathbb{N}$ contains a non-empty open subset in the Polish topology of $\mathbb{Q}^\mathbb{N}$. \square

The argument of Theorem 5.11 cannot be extended to suborders of 2^α for $\omega + 1 \leq \alpha < \omega_1$, since on 2^α the order topology and the product topology do not coincide. In fact, for $\omega + 1 \leq \alpha < \omega_1$, there are more order-preserving functions from 2^α into itself than Borel maps with respect to the product topology. To see this, for every $A \subseteq 2^\mathbb{N}$ consider the function $\varphi_A : 2^\alpha \rightarrow 2^{\omega+1}$ defined by letting

$$\varphi_A(x) = \begin{cases} (x \restriction \omega)^\frown 0 & \text{if } x \restriction \omega \in A; \\ (x \restriction \omega)^\frown 1 & \text{if } x \restriction \omega \notin A. \end{cases}$$

The function φ_A is order-preserving, and $\varphi_A \neq \varphi_B$ whenever $A \neq B$. Since $2^{\omega+1}$ is isomorphic to a suborder of 2^α , this shows that there are at least $2^{2^{\aleph_0}}$ order-preserving functions from 2^α into itself.

5.2. Beyond ZFC. In contrast with the negative results that we showed so far, we now build an uncountable strongly surjective order under extra set theoretic assumptions. For κ an infinite cardinal less than the continuum, consider the statement

BA_κ : up to isomorphism, there is a unique κ -dense suborder of \mathbb{R} .

We know that BA_{\aleph_0} holds in ZFC, while the consistency of BA_{\aleph_1} with ZFC was proved in [B73]. Moreover, BA_{\aleph_1} follows from PFA. The interest for BA_{\aleph_2} was rekindled recently, as witnessed by [MT]. Itay Neeman recently announced a proof of the consistency of BA_{\aleph_2} from large cardinals.

Theorem 5.13. *Let κ be an uncountable cardinal smaller than the continuum. Assume BA_κ . Then there exist strongly surjective orders of cardinality κ .*

We prove in fact the following.

Proposition 5.14. *Let κ be an uncountable cardinal smaller than the continuum. Suppose that, up to isomorphism, X is the unique κ -dense suborder of \mathbb{R} . Then $L \leq_s X$ for any $L \subseteq \mathbb{R}$ with $|L| \leq \kappa$.*

Proof. Let $M \subseteq L$ be a countable dense set in the order topology of L containing all points having an immediate successor or an immediate predecessor in L and the endpoints of L if they exist. Let $Y = \sum_{l \in L} Y_l$, where Y_l is isomorphic to X if $l \in M$, and a singleton otherwise. It is then enough to show that Y is isomorphic to X . It is easily checked that Y is κ -dense, so it remains to check that $Y \leq_i \mathbb{R}$.

For each $l \in M$ let $Z_l \simeq \mathbb{R}$ be a linear order containing Y_l . If $l \in L \setminus M$, let $Z_l = Y_l$. Finally, let Z be the completion of $\sum_{l \in L} Z_l$. The proof will be concluded by showing that $Z \simeq \mathbb{R}$.

By construction, Z does not have minimum nor maximum and it is complete. To apply the classical characterization of the order type of \mathbb{R} ([R82, Theorem 2.30]) it remains to show that Z is separable. For each $l \in M$, pick a countable dense subset Q_l of Z_l . Then $\sum_{l \in M} Q_l$ is dense in Z . \square

Applying Corollary 2.14, one obtains that if X is the order provided by BA_κ , then each X^n is a strongly surjective order. These are in fact distinct order types, more precisely the following holds.

Proposition 5.15. *Assume BA_κ and let X witness it. Then*

$$X <_i X^2 <_i X^3 <_i \dots \text{ and } X <_s X^2 <_s X^3 <_s \dots$$

Proof. Since $X^n \leq_s X^{n+1}$, it is enough to show inductively $X^{n+1} \not\leq_i X^n$. First notice that $X^2 \not\leq_i X$, because X^2 contains an uncountable family of pairwise disjoint open intervals, while X does not.

Now assume $X^{n+1} \not\leq_i X^n$, and suppose towards a contradiction that $f : X^n \times X^2 \rightarrow X^n \times X$ witnesses $X^{n+2} \leq_i X^{n+1}$. Write $f(a, x) =$

$(g(a, x), h(a, x))$. If for some $a \in X^n$ the map $x \mapsto h(a, x)$ is injective then it witnesses $X^2 \leq_i X$, contradicting the base step of our induction. Hence for all $a \in X^n$ there exist distinct $x, y \in X^2$ such that $h(a, x) = h(a, y)$. Fix $a \in X^n$, pick $x = (x_0, x_1)$ and $y = (y_0, y_1)$ with this property, and such that $x <_{X^2} y$. For all $z \in (x, y)_{X^2}$ and $b \in X^n$ we have $(a, x) <_{X^{n+2}} (b, z) <_{X^{n+2}} (a, y)$ and hence $h(b, z) = h(a, x)$. This implies that $(b, z) \mapsto g(b, z)$ is injective and witnesses $X^n \times (x, y)_{X^2} \leq_i X^n$. Now notice that $X \leq_i (x, y)_{X^2}$: this is clear if $x_1 <_X y_1$, while when $x_1 = y_1$ it follows from the fact that $(x_0, y_0)_X$ is κ -dense, and hence isomorphic to X . Putting all together, we have shown that $X^{n+1} \leq_i X^n$, against the induction hypothesis. \square

In Theorem 5.13 we proved the existence of uncountable strongly surjective orders under a consequence of PFA. So it is natural to try to build strongly surjective orders under quite orthogonal principles, like \diamond and its variations. To this end it appears that the following notions, with roots in [B82], are relevant.

Definition 5.16. A partial order (T, \leq_T) (often denoted only by T) is a *tree* if for all $t \in T$ the initial interval $(\leftarrow, t]_T$ has order-type an ordinal α called the *length* of t . The set of nodes of length α is the α *th level* of T : we denote it by T^α . The *height* of T is the smallest ordinal α such that T^α is empty. Moreover we say that $s, t \in T$ belong to the same *brotherhood* of T if $(\leftarrow, t]_T = (\leftarrow, s]_T$.

If every brotherhood of T is linearly ordered, then we can order T by the lexicographic order we denote by \leq_{lex} and, following Baumgartner [B82], we say that T is *doubly ordered*.

Two doubly ordered trees T and S are isomorphic (and we write $(T, \leq_T, \leq_{lex}) \cong (S, \leq_S, \leq_{lex})$) if there exists a bijection $f : T \rightarrow S$ which preserves both the partial and the linear orders.

Definition 5.17. A doubly ordered tree (T, \leq_T, \leq_{lex}) is a *Baumgartner tree* if the following conditions hold:

- (T, \leq_T) is a *Suslin tree* (that is, every chain and every antichain in T is countable, the height of T is ω_1 , and for all $t \in T^\alpha$ and all $\beta < \omega_1$ with $\alpha \leq \beta$ there exists $s \in T^\beta$ such that $t \leq_T s$).
- T has *rational brotherhoods*, that is the ordering of each brotherhood of T is isomorphic to \mathbb{Q} .
- for every $X \in [T]^{\aleph_1}$, if
 - X is cofinal with respect to \leq_T in $\{t \in T \mid \exists s \in X (s \leq_T t)\}$,
 - X has a rational basis, that is the ordering of its minimal elements is isomorphic to \mathbb{Q} ,
then we have $(X, \leq_T, \leq_{lex}) \cong (T, \leq_T, \leq_{lex})$.

Assuming \diamond^+ , in [B82, Theorem 4.15] Baumgartner claimed to build a minimal Specker type which was in fact the linear part of what we

call a Baumgartner tree. As pointed out by Hossein Lamei Ramandi, Baumgartner's proof has however a gap: there are indeed many counterexamples to the crucial Lemma 4.14, stated without proof.

Recently, Daniel Soukup told us that the proof of [HNS90, Theorem 2.3] can be adapted to yield a Baumgartner tree under \diamond^+ .

Theorem 5.18. *The linear order of a Baumgartner tree is strongly surjective.*

Proof. Fix a Baumgartner tree (T, \leq_T, \leq_{lex}) and let $L = (T, \leq_{lex})$.

Observe first that, as T has rational brotherhoods, $(T^0, \leq_{lex}) \simeq \mathbb{Q}$ holds. Hence L can be written as a \mathbb{Q} -sum and $\mathbb{Q} \leq_s L$. For any countable linear order K we have $K \leq_s \mathbb{Q}$ and hence $K \leq_s L$.

It remains to deal with the uncountable suborders of L . Given $X \subseteq T$ uncountable let $K = (X, \leq_{lex})$. Consider

$$X^\circ = \{s \in X \mid X \text{ is cofinal in } \{t \in T \mid s \leq_T t\}\},$$

and call A the set of its \leq_T -minimal elements. As A is an antichain and T is Suslin, A is countable. Let $B = X \setminus X^\circ$: B is not cofinal in any set of the form $\{t \in T \mid s \leq_T t\}$ and hence, by a well-known property of Suslin trees, it is countable. For $s \in B$ let L_s be the singleton order, and for $s \in A$ let $L_s = 1 + L$. Notice that for $s \in A$ we have $L_s \simeq (\{t \in X \mid s \leq_T t\}, \leq_{lex})$ by the last clause in the definition of Baumgartner tree. The set $\{t \in X \mid s <_T t\}$ has indeed a rational basis C : take $t, t' \in C$ with $t <_{lex} t'$ and call $s' = t \cap t' \geq_T s$. The second clause implies that there are immediate successors $s_0 <_{lex} s_1 <_{lex} s_2 <_{lex} s_3 <_{lex} s_4$ of s' such that $s_1 \leq_T t$ and $s_3 \leq_T t'$. Cofinality gives us then $t_i \geq_T s_i$ in C for $i = 0, 2, 4$, so C is densely ordered with no extremes. As an antichain in T , C is countable.

Thus we have $K \simeq \sum_{s \in A \cup B} L_s$, where the sum is taken according to lexicographic order. Since the 0th level of T is ordered as \mathbb{Q} we have $L \simeq L\mathbb{Q}$, so for any countable linear order M we have, as linear orders: $LM \simeq (L\mathbb{Q})M \simeq L(\mathbb{Q}M) \simeq L\mathbb{Q} \simeq L$. In particular $L_s \simeq 1 + L \leq_s L2 \simeq L$ for every $s \in A$, so we have $\sum_{s \in A \cup B} L_s \leq_s LM$, where $M = (A \cup B, \leq_{lex})$. Since M is countable, $LM \simeq L$. Altogether we showed $K \leq_s L$, as needed. \square

5.3. Some directions for further research. In this paper we have given a fairly complete treatment of countable strongly surjective linear orders. On the other hand, the uncountable ones open various lines for further research. We discuss briefly here some of them.

5.3.1. The existence of uncountable strongly surjective orders. The main open question concerning strongly surjective orders is the following:

Question 1. Does there exist an uncountable strongly surjective order in ZFC?

Let us mention that in a recent preprint ([MR]), Moore and Ramandi build a model of set theory with no minimal non- σ -scattered linear orders. Such a model is a good candidate to show that consistently there are no uncountable strongly surjective linear orders.

A question perhaps easier to answer in the positive is the following:

Question 2. Does there exist an uncountable strongly surjective order under CH? Under \diamond ?

5.3.2. *Definably strongly surjective orders.* In Theorem 5.11 we proved that no uncountable Borel suborder of $2^{\mathbb{N}}$ can be strongly surjective by using definability reasons: such an order cannot surject onto a non-analytic suborder, since epimorphisms are Borel. This suggests that there may be some Borel subsets of $2^{\mathbb{N}}$ for which this is the only obstruction to strong surjectivity, as they admit epimorphisms onto all their analytic suborders. Call such orders *definably strongly surjective*.

Corollaries 5.4 and 5.10 (but not Corollary 5.12) show that $2^{\mathbb{N}}$ and $\mathbb{Z}^{\mathbb{N}}$ ordered lexicographically are not definably strongly surjective.

Question 3. Do there exist definably strongly surjective orders that are not strongly surjective? In particular, is $\mathbb{Q}^{\mathbb{N}}$ definably strongly surjective? Can the concept of a definably strongly surjective order be extended beyond the Borel suborders of $2^{\mathbb{N}}$?

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